

STABILITY CONDITIONS AND BIRATIONAL GEOMETRY OF PROJECTIVE SURFACES

YUKINOBU TODA

ABSTRACT. We show that the minimal model program on any smooth projective surface is realized as a variation of the moduli spaces of Bridgeland stable objects in the derived category of coherent sheaves.

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1. INTRODUCTION

1.1. Motivation. This paper is a continuation of the previous paper [21], in which the following question on the relationship between minimal model program (MMP) and Bridgeland stability conditions [7] was addressed: (cf. [21, Question 1.1])

Question 1.1. *Let X be a smooth projective variety and consider its MMP,*

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_N.$$

Then is each X_i a moduli space of Bridgeland (semi)stable objects in the derived category of coherent sheaves on X , and MMP is interpreted as wall-crossing under a variation of Bridgeland stability conditions?

The main result of [21] was to answer the above question for the first step of MMP, i.e. extremal contraction, when $\dim X \leq 3$. The purpose of this paper is to give a complete answer to the above question for further steps of MMP when $\dim X = 2$.

1.2. Bridgeland stability. For a smooth projective variety X , Bridgeland [7] introduced the notion of stability conditions on $D^b \text{Coh}(X)$, which provides a mathematical framework of Douglas's II-stability [10]. In [7], Bridgeland showed that the set of stability conditions

$$(1) \quad \text{Stab}(X)$$

forms a complex manifold, and studied it when X is a K3 surface or an abelian surface [8]. Since then there have been several studies on the space (1), or the associated moduli spaces of semistable objects in the derived category, when X is a K3 surface or an abelian surface. (cf. [13], [1], [22], [18], [19], [25].)

On the other hand, there are few literatures in which the space (1) is studied for an arbitrary projective surface X . If X is non-minimal, the birational geometry of X is interesting, and we expect that it has a deep connection with the space of stability conditions (1). This idea is motivated by Bridgeland's work [6] on the construction of three dimensional flops as moduli spaces of objects in the derived category. This result is not yet possible to realize in terms of Bridgeland stability conditions, since constructing them on projective 3-folds turned out to be a very difficult problem. (cf. [4].) However in the surface case, we have the examples of stability conditions constructed by Arcara-Bertram [1]. Given the above background, we shall establish a rigorous statement connecting two dimensional MMP and the space of Bridgeland stability conditions (1).

1.3. Main result. Our main result is formulated in the space $\text{Stab}(X)_{\mathbb{R}}$, defined to be the 'real part' of the space (1). This is the space which fits into the Cartesian square (cf. Section 2)

$$(2) \quad \begin{array}{ccc} \text{Stab}(X)_{\mathbb{R}} & \longrightarrow & \text{Stab}(X) \\ \Pi_{\mathbb{R}} \downarrow & \square & \downarrow \Pi \\ \text{NS}(X)_{\mathbb{R}} & \xrightarrow{-\int_X e^{-i*}} & N(X)_{\mathbb{C}}^{\vee}. \end{array}$$

Recall that the ample cone $A(X) \subset \text{NS}(X)_{\mathbb{R}}$ plays an important role in birational geometry. We will see that there is an open subset

$$(3) \quad U(X) \subset \text{Stab}(X)_{\mathbb{R}}$$

which is homeomorphic to $A(X)$ under the map $\Pi_{\mathbb{R}}$ of the diagram (2). The subset (3) coincides with the set of $\sigma \in \text{Stab}(X)_{\mathbb{R}}$ in which all the objects \mathcal{O}_x for $x \in X$ are stable. The closure $\overline{U}(X)$ is the analogue of the nef cone of X , and expected to contain information of birational geometry of X .

Our purpose is to construct an open subset such as (3) associated to each birational morphism $f: X \rightarrow Y$, and investigate how they are related under the change of (f, Y) . Here we fix the notation: for a Bridgeland stability condition $\sigma = (Z, \mathcal{A})$, we denote by $\mathcal{M}^{\sigma}([\mathcal{O}_x])$

the algebraic space which parameterizes Z -stable objects $E \in \mathcal{A}$ with phase one and $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ for $x \in X$. (cf. [14].) The following is the main theorem in this paper:

Theorem 1.2. (Proposition 4.11, Proposition 4.13) *Let X be a smooth projective complex surface. Then for any smooth projective surface Y and a birational morphism $f: X \rightarrow Y$, there is a connected open subset*

$$U(Y) \subset \text{Stab}(X)_{\mathbb{R}}$$

satisfying the following conditions:

- *If f factors through a blow-up at a point $Y' \rightarrow Y$, we have*

$$(4) \quad \overline{U}(Y) \cap \overline{U}(Y') \neq \emptyset,$$

which is real codimension one in $\text{Stab}(X)_{\mathbb{R}}$.

- *For any $\sigma \in U(Y)$, $\mathcal{M}^{\sigma}([\mathcal{O}_x])$ is isomorphic to Y .*

The above result shows that the space $\text{Stab}(X)_{\mathbb{R}}$ is a fundamental object, beyond the ample cone $A(X)$, in the study of birational geometry of X . Indeed, the geometry of any birational morphism $f: X \rightarrow Y$ is captured from the space $\text{Stab}(X)_{\mathbb{R}}$. The following is the obvious corollary of Theorem 1.2:

Corollary 1.3. *Let X be a smooth projective complex surface and*

$$X = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_N$$

a MMP, i.e. contractions of (-1) -curves. Then there is a continuous one parameter family of Bridgeland stability conditions $\{\sigma_t\}_{t \in (0,1)}$ on $D^b \text{Coh}(X)$ and real numbers

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$$

such that X_i is isomorphic to $\mathcal{M}^{\sigma_t}([\mathcal{O}_x])$ for $t \in (t_{i-1}, t_i)$.

The result of the above corollary completely answers Question 1.1 for surfaces: any MMP of a smooth projective surface X is realized as wall-crossing of Bridgeland moduli spaces of stable objects in $D^b \text{Coh}(X)$. The real numbers t_i correspond to walls in this story.

1.4. Technical ingredients. As we mentioned before, the space (1) has not been studied for an arbitrary projective surface X . Although it was studied for a K3 surface or an abelian surface in [8], there are several technical arguments in [8] which are not applied directly to an arbitrary projective surface X . It seems that these technical issues have prevented us to study the space (1) beyond the case of K3 surfaces or abelian surfaces.

One of the technical issues is to prove the support property of the stability conditions. This property is required in order to make the topology of the space (1) desirable. It is now turned out that proving

the support property is not an easy problem in general, and closely related to the Bogomolov-Gieseker (BG) type inequality of semistable objects in the derived category. In the case of K3 surfaces or abelian surfaces, proving the BG type inequality is easier: this follows from the Serre duality and the Riemann-Roch theorem. However this is not the case for an arbitrary projective surface, and we need to find a general argument proving such an inequality. In the previous paper [21], we established such a BG type inequality for semistable objects on an arbitrary projective surface, and proved the support property for some stability conditions in $\overline{U}(X)$. We use this result to show the support property for stability conditions contained in other subsets $\overline{U}(Y)$.

Another issue is that the analysis of the boundary of $U(X)$ in the case of K3 surfaces in [7] is not applied for an arbitrary projective surface X . In the former case, if we cross the codimension one boundary of $U(X)$, then the resulting stability condition is obtained by applying some autoequivalence of the derived category. In the latter case, this is not the case in general. Indeed we will see that, after crossing the boundary of $U(X)$ corresponding to a (-1) -curve contraction, then the resulting stability condition is not described by an autoequivalence but by a certain tilting of the t-structure which appears at the boundary. We will describe the resulting tilting explicitly, and investigate the wall-crossing behavior of the open subsets $U(Y)$ in Theorem 1.2 in detail.

1.5. Relation to existing works. There are some recent works in which the relationship between Bridgeland stability conditions and MMP is discussed. (cf. [2], [3], [23], [21].) The works [2], [3] treat the cases of \mathbb{P}^2 and K3 surfaces respectively. Also the works [23], [21] treat the cases of local flops, contraction of a (-1) -curve, respectively. The result in this paper generalizes the result of [21], and completely answer [21, Question 1.1] for an arbitrary projective surface.

The examples of Bridgeland stability conditions on arbitrary projective surfaces are given in [1]. In the works [18], [19], [25], [16], [17], the structure of walls and wall-crossing phenomena with respect to these stability conditions are studied. Our construction of $U(Y)$ provides another examples of Bridgeland stability conditions on arbitrary non-minimal surfaces. It would be interesting to study the moduli spaces of semistable objects in $U(Y)$ with arbitrary numerical classes, and investigate their behavior under crossing the intersection of the closures (4).

1.6. Plan of the paper. In Section 2, we give some background on Bridgeland stability conditions, especially on projective surfaces. In Section 3, we construct some t-structures on relevant triangulated categories. In Section 4, we give a proof of Theorem 1.2. In Section 5, we prove some technical results which are stated in previous sections.

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1.8. Notation and convention. In this paper, all the varieties are defined over \mathbb{C} . For a triangulated category \mathcal{D} and a set of objects $\mathcal{S} \subset \mathcal{D}$, we denote by $\langle \mathcal{S} \rangle$ the smallest extension closed subcategory of \mathcal{D} which contains objects in \mathcal{S} . The category $\langle \mathcal{S} \rangle$ is called the extension closure of \mathcal{S} . For the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$, we denote by $\mathcal{H}_{\mathcal{A}}^i(*)$ the i -th cohomology functor with respect to the t-structure with heart \mathcal{A} . If \mathcal{S} is contained in \mathcal{A} , the right orthogonal complement of \mathcal{S} in \mathcal{A} is defined by

$$\mathcal{S}^{\perp} := \{E \in \mathcal{A} : \text{Hom}(\mathcal{S}, E) = 0\}.$$

2. BACKGROUND

In this section, we briefly recall Bridgeland stability conditions, and prepare some results which will be needed in the later sections.

2.1. Bridgeland stability conditions. Let X be a smooth projective variety and $N(X)$ the numerical Grothendieck group of X . This is the quotient of the usual Grothendieck group $K(X)$ by the subgroup of $E \in K(X)$ with $\chi(E, F) = 0$ for any $F \in K(X)$, where $\chi(E, F)$ is the Euler pairing

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(E, F).$$

Definition 2.1. ([7]) *A stability condition on X is a pair*

$$(5) \quad (Z, \mathcal{A}), \quad \mathcal{A} \subset D^b \text{Coh}(X),$$

where $Z: N(X) \rightarrow \mathbb{C}$ is a group homomorphism and \mathcal{A} is the heart of a bounded t-structure, such that the following conditions hold:

- *For any non-zero $E \in \mathcal{A}$, we have*

$$(6) \quad Z(E) \in \{r \exp(i\pi\phi) : r > 0, \phi \in (0, 1]\}.$$

- *(Harder-Narasimhan property) For any $E \in \mathcal{A}$, there is a filtration in \mathcal{A}*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N$$

such that each subquotient $F_i = E_i/E_{i-1}$ is Z -semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here an object $E \in \mathcal{A}$ is Z -(semi)stable if for any subobject $0 \neq F \subsetneq E$ we have

$$\arg Z(F) < (\leq) \arg Z(E).$$

The group homomorphism Z is called a *central charge*. The central charges we use in this paper are of the form

$$(7) \quad Z_\omega(E) = - \int_X e^{-i\omega} \operatorname{ch}(E),$$

for $\omega \in \operatorname{NS}(X)_\mathbb{R}$. If $\dim X = 2$, we have

$$(8) \quad Z_\omega(E) = -\operatorname{ch}_2(E) + \frac{\omega^2}{2} \operatorname{ch}_0(E) + i \operatorname{ch}_1(E) \cdot \omega.$$

We fix a norm $\|\cdot\|$ on the finite dimensional vector space $N(X)_\mathbb{R}$. We need to put the following technical condition on the stability conditions:

Definition 2.2. *A stability condition (5) satisfies the support property if there is a constant $K > 0$ such that for any non-zero Z -semistable object $E \in \mathcal{A}$, we have*

$$\frac{\|E\|}{|Z(E)|} < K.$$

The set $\operatorname{Stab}(X)$ is defined to be the set of stability conditions on $D^b \operatorname{Coh}(X)$ satisfying the support property. The following is the main result of [7]. (Also see [15].)

Theorem 2.3. ([7]) *There is a natural topology on $\operatorname{Stab}(X)$ such that the forgetting map*

$$\Pi: \operatorname{Stab}(X) \rightarrow N(X)_\mathbb{C}^\vee$$

sending (Z, \mathcal{A}) to Z is a local homeomorphism.

We are interested in the set of stability conditions whose central charges are of the form (7). So we restrict our attention to the space $\operatorname{Stab}(X)_\mathbb{R}$ defined as follows:

Definition 2.4. *We define $\operatorname{Stab}(X)_\mathbb{R}$ to be the Cartesian square*

$$(9) \quad \begin{array}{ccc} \operatorname{Stab}(X)_\mathbb{R} & \longrightarrow & \operatorname{Stab}(X) \\ \Pi_\mathbb{R} \downarrow & \square & \downarrow \Pi \\ \operatorname{NS}(X)_\mathbb{R} & \xrightarrow{-\int_X e^{-i*}} & N(X)_\mathbb{C}^\vee. \end{array}$$

Here the bottom map takes $\omega \in \operatorname{NS}(X)_\mathbb{R}$ to the central charge Z_ω given by (7).

2.2. Gluing t-structures. We use the following *gluing t-structure* method in order to produce several t-structures. Let

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{j} \mathcal{E}$$

be an *exact triple* of triangulated categories. Namely \mathcal{C} , \mathcal{D} and \mathcal{E} are triangulated categories, i, j are exact functors with $j \circ i = 0$. Both of i and j have the left and the right adjoint functors, which satisfy some axioms. For the detail, see [11, IV. Ex. 2].

Let

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}), \quad (\mathcal{E}^{\leq 0}, \mathcal{E}^{\geq 0}),$$

be bounded t-structures on \mathcal{C} and \mathcal{E} respectively. Then they induce the bounded t-structure on \mathcal{D} whose heart is given by

$$\{E \in \mathcal{D} : j(E) \in \mathcal{E}^0, \text{Hom}(i(\mathcal{C}^{<0}), E) = \text{Hom}(E, i(\mathcal{C}^{>0})) = 0\}.$$

Here $\mathcal{E}^0 := \mathcal{E}^{\leq 0} \cap \mathcal{E}^{\geq 0}$ is the heart on \mathcal{E} . For the detail, see [5, n. 1.4], [11, IV. Ex. 4].

2.3. Perverse t-structure. Let X and Y be smooth projective surfaces, and f a birational morphism

$$f : X \rightarrow Y.$$

We recall the construction of the perverse t-structure associated to the above data, following [6], [9].

It is well-known that the derived pull-back

$$\mathbf{L}f^* : D^b \text{Coh}(Y) \rightarrow D^b \text{Coh}(X)$$

is fully-faithful. The functor $\mathbf{L}f^*$ has the right adjoint $\mathbf{R}f_*$ and the left adjoint $\mathbf{R}f_!$,

$$\mathbf{R}f_*, \mathbf{R}f_! : D^b \text{Coh}(X) \rightarrow D^b \text{Coh}(Y)$$

where $\mathbf{R}f_!$ is given by

$$\mathbf{R}f_! E = \mathbf{R}f_*(E \otimes \omega_X) \otimes \omega_Y^{-1}.$$

We define the triangulated subcategories $\mathcal{C}_{X/Y}$, $\mathcal{D}_{X/Y}$ in $D^b \text{Coh}(X)$ to be

$$\begin{aligned} \mathcal{C}_{X/Y} &:= \{E \in D^b \text{Coh}(X) : \mathbf{R}f_! E \cong 0\}, \\ \mathcal{D}_{X/Y} &:= \{E \in D^b \text{Coh}(X) : \mathbf{R}f_* E \cong 0\}. \end{aligned}$$

They are related by $\mathcal{C}_{X/Y} \otimes \omega_X = \mathcal{D}_{X/Y}$. Here we only use the latter category $\mathcal{D}_{X/Y}$. The category $\mathcal{C}_{X/Y}$ will be treated in the next section.

We have the sequences of exact functors

$$\mathcal{D}_{X/Y} \rightarrow D^b \text{Coh}(X) \xrightarrow{\mathbf{R}f_*} D^b \text{Coh}(Y),$$

where the left functor is the natural inclusion. The above sequence determines an exact triple, and the standard t-structure on $D^b \text{Coh}(X)$ induces a t-structure

$$(\mathcal{D}_{X/Y}^{\leq 0}, \mathcal{D}_{X/Y}^{\geq 0})$$

on $\mathcal{D}_{X/Y}$. (cf. [6, Lemma 3.1].) By gluing the shifted t-structure $(\mathcal{D}_{X/Y}^{\leq -1}, \mathcal{D}_{X/Y}^{\geq -1})$ and the standard t-structure on $D^b \text{Coh}(Y)$, we have the heart of the perverse t-structure (cf. [6], [9])

$$\text{Per}(X/Y) \subset D^b \text{Coh}(X).$$

The perverse heart $\text{Per}(X/Y)$ is known to be equivalent to the module category of a certain sheaf of non-commutative coherent \mathcal{O}_Y -algebras. (cf. [9].) In particular, it is a noetherian abelian category. Also if $f = \text{id}_X: X \rightarrow X$, the category $\text{Per}(X/X)$ coincides with $\text{Coh}(X)$.

2.4. Tilting of $\text{Per}(X/Y)$. Let us take

$$\omega \in \text{NS}(Y)_{\mathbb{Q}},$$

such that ω is a \mathbb{Q} -ample class. We have the following slope function,

$$\mu_{f^*\omega}: \text{Per}(X/Y) \setminus \{0\} \rightarrow \mathbb{Q} \cup \{\infty\},$$

by setting $\mu_{f^*\omega}(E) = \infty$ if $\text{ch}_0(E) = 0$, and

$$\mu_{f^*\omega}(E) = \frac{\text{ch}_1(E) \cdot f^*\omega}{\text{ch}_0(E)},$$

if $\text{ch}_0(E) \neq 0$. The above slope function determines a weak stability condition on $\text{Per}(X/Y)$, which satisfies the Harder-Narasimhan property. (cf. [21, Lemma 3.6].)

We define the pair of subcategories $(\mathcal{T}_{f^*\omega}, \mathcal{F}_{f^*\omega})$ in $\text{Per}(X/Y)$ to be

$$\begin{aligned} \mathcal{T}_{f^*\omega} &:= \langle E : E \text{ is } \mu_{f^*\omega}\text{-semistable with } \mu_{f^*\omega}(E) > 0 \rangle, \\ \mathcal{F}_{f^*\omega} &:= \langle E : E \text{ is } \mu_{f^*\omega}\text{-semistable with } \mu_{f^*\omega}(E) \leq 0 \rangle. \end{aligned}$$

The above pair is a torsion pair [12] in $\text{Coh}(X)$. The associated tilting is

$$(10) \quad \mathcal{A}_{f^*\omega} := \langle \mathcal{F}_{f^*\omega}[1], \mathcal{T}_{f^*\omega} \rangle.$$

By a general theory of tilting, the category $\mathcal{A}_{f^*\omega}$ is the heart of a bounded t-structure on $D^b \text{Coh}(X)$. In particular, it is an abelian category. Later we will need the following property on the above category.

Lemma 2.5. *We have the embedding*

$$\mathbf{L}f^* \mathcal{A}_{\omega} \subset \mathcal{A}_{f^*\omega}.$$

Proof. It is enough to show the following statements:

- For any $M \in \text{Coh}(Y)$, we have $\mathbf{L}f^* M \in \text{Per}(X/Y)$.
- If M is a torsion free μ_{ω} -semistable sheaf on Y , then $\mathbf{L}f^* M \in \text{Per}(X/Y)$ is $\mu_{f^*\omega}$ -semistable.

We first show the first statement. By the projection formula, we have

$$\mathbf{R}f_* \mathbf{L}f^* M \cong M.$$

Also we have

$$\mathrm{Hom}(\mathbf{L}f^* M, \mathcal{D}_{X/Y}^{\geq 0}) \cong 0$$

by adjunction. Let us take $F \in \mathcal{D}_{X/Y}^{\leq -2}$. We have

$$\begin{aligned} \mathrm{Hom}(F, \mathbf{L}f^* M) &\cong \mathrm{Hom}(\mathbf{R}f_! F, M) \\ &\cong 0 \end{aligned}$$

since $\mathbf{R}f_! F \in \mathrm{Coh}^{\leq -1}(Y)$. Therefore $\mathbf{L}f^* M \in \mathrm{Per}(X/Y)$ follows by the definition of the gluing.

As for the second statement, let us take an exact sequence in $\mathrm{Per}(X/Y)$

$$0 \rightarrow F \rightarrow \mathbf{L}f^* M \rightarrow G \rightarrow 0,$$

such that F and G are non-zero. We need to show that

$$(11) \quad \mu_{f^*\omega}(F) \leq \mu_{f^*\omega}(G).$$

Applying $\mathbf{R}f_*$, we obtain the exact sequence in $\mathrm{Coh}(Y)$

$$0 \rightarrow \mathbf{R}f_* F \rightarrow M \rightarrow \mathbf{R}f_* G \rightarrow 0.$$

If both of $\mathbf{R}f_*(F)$ and $\mathbf{R}f_*(G)$ are non-zero, the inequality (11) holds by the μ_ω -stability of E and noting $\mu_{f^*\omega}(\mathbf{L}f^*(*)) = \mu_\omega(*)$ for non-zero $*$. If $\mathbf{R}f_* G = 0$, then $\mu_{f^*\omega}(G) = \infty$ and (11) holds. Suppose that $\mathbf{R}f_* F = 0$. Then $F \in \mathcal{D}_{X/Y} \cap \mathrm{Coh}(X)[1]$, hence $\mathbf{R}f_! F \in D^{\leq 0}(\mathrm{Coh}(Y))$ and its zero-th cohomology is a zero dimensional sheaf. By adjunction and the torsion freeness of M , this implies

$$\begin{aligned} \mathrm{Hom}(F, \mathbf{L}f^* M) &\cong \mathrm{Hom}(\mathbf{R}f_! F, M) \\ &\cong 0, \end{aligned}$$

which is a contradiction. \square

2.5. Bridgeland stability conditions on projective surfaces. Let $f: X \rightarrow Y$ be a birational morphism between smooth projective surfaces, and $\omega \in \mathrm{NS}(Y)_{\mathbb{Q}}$ is ample. We consider the pair

$$\sigma_{f^*\omega} := (Z_{f^*\omega}, \mathcal{A}_{f^*\omega}),$$

where $Z_{f^*\omega}: N(X) \rightarrow \mathbb{C}$ is the central charge defined by (8), and $\mathcal{A}_{f^*\omega}$ is the heart of a bounded t-structure on $D^b \mathrm{Coh}(X)$ constructed in the previous subsection. We have the following proposition.

Proposition 2.6. *Suppose that f satisfies one of the following conditions:*

- $f = \mathrm{id}_X: X \rightarrow X$.
- f contracts a single (-1) -curve C on X to a point in Y .

Then we have

$$\sigma_{f^*\omega} \in \text{Stab}(X)_{\mathbb{R}}.$$

In particular, $\sigma_{f^*\omega}$ satisfies the support property.

Proof. If $f = \text{id}_X$, the result of [1] shows that $\sigma_{f^*\omega}$ is a stability condition on $D^b \text{Coh}(X)$. If f contracts a (-1) -curve C on X , the result of [21, Lemma 3.12] shows that $\sigma_{f^*\omega}$ is a stability condition on $D^b \text{Coh}(X)$. The support property of $\sigma_{f^*\omega}$ is proven in [21, Proposition 3.13] when f contracts a (-1) -curve. When $f = \text{id}_X$, the proof for the support property follows from the same (even easier) argument of [21, Proposition 3.13]. \square

If $f = \text{id}_X$, the stability condition σ_{ω} satisfies the following property:

Lemma 2.7. *Let $\omega \in \text{NS}(X)_{\mathbb{Q}}$ be ample and $f = \text{id}_X$.*

(i) *For any $x \in X$, the object \mathcal{O}_x is a simple object in \mathcal{A}_{ω} . In particular, it is Z_{ω} -stable.*

(ii) *For any object $E \in \mathcal{A}_{\omega}$ with $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$, we have $E \cong \mathcal{O}_x$ for some $x \in X$.*

Proof. The result of (i) is essentially proved in [8, Lemma 6.3]. The result of (ii) is obvious from the construction of \mathcal{A}_{ω} . \square

The ample cone $A(X)$ is defined to be

$$A(X) := \{\omega \in \text{NS}(X)_{\mathbb{R}} : \omega \text{ is } \mathbb{R}\text{-ample}\}.$$

We define its partial compactification $\overline{A}(X) \subset \text{NS}(X)_{\mathbb{R}}$ to be

$$\overline{A}(X) := \bigcup_{f: X \rightarrow Y} f^* A(Y).$$

In the above union, f is either $\text{id}_X: X \rightarrow X$ or contracts a single (-1) -curve on X to a point in Y . Below, we sometimes write an element of $\overline{A}(X)$ as ω for a nef divisor ω on X , omitting f^* in the notation. We have the embedding

$$(12) \quad \overline{A}(X) \subset \text{NS}(X)_{\mathbb{R}}.$$

The following proposition shows the existence of stability conditions for irrational ω :

Proposition 2.8. *The embedding (12) lifts to a continuous map*

$$(13) \quad \sigma: \overline{A}(X) \rightarrow \text{Stab}(X)_{\mathbb{R}}$$

which takes any rational point $\omega \in \overline{A}(X)$ to the stability condition σ_{ω} in Proposition 2.6.

Proof. The proof will be given in Subsection 5.1. \square

Remark 2.9. For $\omega \in A(X)$, it is possible to construct the heart \mathcal{A}_ω similarly to (10), even if ω is irrational. However the Harder-Narasimhan property for the pair $(Z_\omega, \mathcal{A}_\omega)$ is not obvious. In the proof of Proposition 2.8, we will also show that any object \mathcal{O}_x for $x \in X$ is $\sigma(\omega)$ -stable, even when ω is irrational. Combined with Lemma 5.1, it is shown that the pair $(Z_\omega, \mathcal{A}_\omega)$ indeed satisfies the Harder-Narasimhan property for an irrational ω . (Also see the argument of [8, Section 11].)

We set $U(X) \subset \text{Stab}(X)_\mathbb{R}$ to be

$$U(X) := \sigma(A(X))$$

Note that $U(X)$ is a connected open subset of $\text{Stab}(X)_\mathbb{R}$, which is homeomorphic to $A(X)$ under the forgetting map $\text{Stab}(X)_\mathbb{R} \rightarrow \text{NS}(X)_\mathbb{R}$. It satisfies the property of Theorem 1.2 for $f = \text{id}_X: X \rightarrow X$. Our purpose in the following sections is to construct a similar open subset associated to any birational morphism $f: X \rightarrow Y$.

3. CONSTRUCTION OF T-STRUCTURES

In what follows, X and Y are smooth projective surfaces and

$$f: X \rightarrow Y$$

a birational morphism. In this section, we construct some t-structures on $\mathcal{C}_{X/Y}$ and $D^b \text{Coh}(X)$, which will be needed in the proof of Theorem 1.2.

3.1. t-structure on $\mathcal{C}_{X/Y}$. Let $\mathcal{C}_{X/Y}$ be the triangulated subcategory of $D^b \text{Coh}(X)$ defined in Subsection 2.3. The purpose here is to construct the heart of a bounded t-structure

$$\mathcal{C}_{X/Y}^0 \subset \mathcal{C}_{X/Y}$$

satisfying the following conditions: there are objects $S_1, \dots, S_N \in \mathcal{C}_{X/Y}^0$ satisfying

$$(14) \quad \mathcal{C}_{X/Y}^0 = \langle S_1, \dots, S_N \rangle, \quad \mathbf{R}f_* S_i[1] \in \text{Coh}_0(Y).$$

Here $\text{Coh}_0(Y)$ is the abelian category of zero dimensional coherent sheaves on Y , and N is the number of irreducible components of $\text{Ex}(f)$, the exceptional locus of f .

It will turn out later (Corollary 3.8) that the heart $\mathcal{C}_{X/Y}^0$ is a certain tilting of the standard heart $\mathcal{C}_{X/Y} \cap \text{Coh}(X)$. However our definition here is rather indirect: we construct $\mathcal{C}_{X/Y}^0$ by the induction on the number of irreducible components N . This latter approach is more convenient to describe generators of $\mathcal{C}_{X/Y}^0$, and the relationship under blow-downs.

When $N = 0$, then $\mathcal{C}_{X/Y} = \{0\}$ and the heart $\mathcal{C}_{X/Y}^0$ is taken to be the trivial one. Suppose that $N > 0$, and let us consider the finite set of points,

$$f(\text{Ex}(f)) = \{p_1, \dots, p_l\}.$$

If $l > 1$, then any object $E \in \mathcal{C}_{X/Y}$ uniquely decomposes as

$$E \cong E_1 \oplus \dots \oplus E_l,$$

where E_i is supported on $f^{-1}(p_i)$. Since each E_i is an object in \mathcal{C}_{X/Y_i} for some birational morphism $f_i: X \rightarrow Y_i$ satisfying $\sharp f_i(\text{Ex}(f_i)) = 1$, we may assume that $l = 1$.

Let

$$h: Y' \rightarrow Y$$

be the blowing up at $f(\text{Ex}(f)) = \{p\}$, and $C \subset Y'$ the exceptional locus of h . The birational morphism $f: X \rightarrow Y$ factors through $h: Y' \rightarrow Y$,

$$f: X \xrightarrow{g} Y' \xrightarrow{h} Y.$$

The functor $\mathbf{R}f_!$ also factors as

$$\mathbf{R}f_!: D^b \text{Coh}(X) \xrightarrow{\mathbf{R}g_!} D^b \text{Coh}(Y') \xrightarrow{\mathbf{R}h_!} D^b \text{Coh}(Y).$$

Therefore we have the sequence of exact functors

$$(15) \quad \mathcal{C}_{X/Y'} \rightarrow \mathcal{C}_{X/Y} \xrightarrow{\mathbf{R}g_!} \mathcal{C}_{Y'/Y},$$

where the left functor is the natural inclusion. The functors $\mathbf{L}g^*, g^!$ satisfy

$$\mathbf{R}g_! \mathbf{L}g^* E \cong E, \quad \mathbf{R}g_! g^! E \cong E.$$

This implies that $\mathbf{L}g^*$ and $g^!$ induce the right and the left adjoint functors of

$$\mathbf{R}g_!: \mathcal{C}_{X/Y} \rightarrow \mathcal{C}_{Y'/Y}$$

respectively. From this fact, it is straightforward to check that the sequence (15) is an exact triple as in Subsection 2.2.

By [6, Lemma 3.1], the standard t-structure on $D^b \text{Coh}(Y')$ induces a bounded t-structure on $\mathcal{C}_{Y'/Y}$. The heart is described by (cf. [9, Proposition 3.5.8])

$$(16) \quad \mathcal{C}_{Y'/Y} \cap \text{Coh}(Y') = \langle \mathcal{O}_C \rangle.$$

On the other hand, by the assumption of the induction, we have the heart $\mathcal{C}_{X/Y'}^0 \subset \mathcal{C}_{X/Y'}$ written as

$$(17) \quad \mathcal{C}_{X/Y'}^0 = \langle S'_1, \dots, S'_{N-1} \rangle,$$

for some objects $S'_j \in \mathcal{C}_{X/Y'}^0$ with $1 \leq j \leq N-1$ satisfying $\mathbf{R}g_* S'_j[1] \in \text{Coh}_0(Y')$. By gluing the t-structures with hearts (16), (17) via the exact triple (15), we obtain the heart

$$\tilde{\mathcal{C}}_{X/Y}^0 \subset \mathcal{C}_{X/Y}.$$

Let us set $\widehat{C} := g^*C$. We have the following lemma:

Lemma 3.1. *We have*

$$(18) \quad \tilde{\mathcal{C}}_{X/Y}^0 = \langle \mathcal{C}_{X/Y'}^0, \mathcal{O}_{\widehat{C}} \rangle,$$

such that $(\mathcal{C}_{X/Y'}^0, \langle \mathcal{O}_{\widehat{C}} \rangle)$ is a torsion pair on $\tilde{\mathcal{C}}_{X/Y}^0$.

Proof. We first check that the RHS is contained in the LHS. By the definition of gluing, it is obvious that $\mathcal{C}_{X/Y'}^0$ is contained in the LHS. Also since $\mathcal{O}_{\widehat{C}} = \mathbf{L}g^*\mathcal{O}_C$, we have $\mathbf{R}g_!\mathcal{O}_{\widehat{C}} = \mathcal{O}_C \in \text{Coh}(Y')$. We have

$$(19) \quad \begin{aligned} \text{Hom}(\mathcal{C}_{X/Y'}, \mathcal{O}_{\widehat{C}}) &\cong \text{Hom}(\mathbf{R}g_!\mathcal{C}_{X/Y'}, \mathcal{O}_C) \\ &\cong 0 \end{aligned}$$

since $\mathbf{R}g_!\mathcal{C}_{X/Y'} = 0$, and

$$\begin{aligned} \text{Hom}(\mathcal{O}_{\widehat{C}}, \mathcal{C}_{X/Y'}^{>0}) &\cong \text{Hom}(\mathcal{O}_C, \mathbf{R}g_*\mathcal{C}_{X/Y'}^{>0}) \\ &\cong 0 \end{aligned}$$

since $\mathbf{R}g_*\mathcal{C}_{X/Y'}^{>0} \in D^{>1}(\text{Coh}(Y))$ by the assumption of the induction. These imply that $\mathcal{O}_{\widehat{C}}$ is contained in the LHS.

Conversely, let us take an object $E \in \tilde{\mathcal{C}}_{X/Y}^0$. By taking the adjunction, we have the distinguished triangle

$$F \rightarrow E \rightarrow \mathbf{L}g^*\mathbf{R}g_!E.$$

Note that we have

$$\mathbf{L}g^*\mathbf{R}g_!E \in \langle \mathcal{O}_{\widehat{C}} \rangle, \quad F \in \mathcal{C}_{X/Y'}.$$

Moreover, since $\mathcal{C}_{X/Y'}^0 \subset \tilde{\mathcal{C}}_{X/Y}^0$, we have

$$\mathcal{H}_{\mathcal{C}_{X/Y'}^0}^i(F) \cong \mathcal{H}_{\tilde{\mathcal{C}}_{X/Y}^0}^i(F)$$

for all i . Therefore we have the exact sequence in $\tilde{\mathcal{C}}_{X/Y}^0$

$$0 \rightarrow \mathcal{H}_{\mathcal{C}_{X/Y'}^0}^0(F) \rightarrow E \rightarrow \mathbf{L}g^*\mathbf{R}g_!E \rightarrow \mathcal{H}_{\mathcal{C}_{X/Y'}^0}^1(F) \rightarrow 0,$$

and $\mathcal{H}_{\mathcal{C}_{X/Y'}^0}^i(F) = 0$ for $i \neq 0, 1$. On the other hand, for any $A \in \text{Coh}(Y')$ and $A' \in \mathcal{C}_{X/Y'}^0$, we have

$$\begin{aligned} \text{Hom}(\mathbf{L}g^*A, A') &\cong \text{Hom}(A, \mathbf{R}g_*A') \\ &\cong 0, \end{aligned}$$

since $\mathbf{R}g_*A' \in \text{Coh}_0(Y')[-1]$. Therefore we have $\mathcal{H}_{\mathcal{C}_{X/Y'}}^1(F) \cong 0$, $F \in \mathcal{C}_{X/Y'}^0$ and an exact sequence in $\tilde{\mathcal{C}}_{X/Y}^0$

$$(20) \quad 0 \rightarrow F \rightarrow E \rightarrow \mathbf{L}g^*\mathbf{R}g_!E \rightarrow 0.$$

This implies that E is contained in the RHS of (18). Together with the vanishing (19), the exact sequence (20) implies that $(\mathcal{C}_{X/Y'}^0, \langle \mathcal{O}_{\hat{C}} \rangle)$ is a torsion pair on $\tilde{\mathcal{C}}_{X/Y}^0$. \square

We also have the following lemma:

Lemma 3.2. *There is a torsion pair on $\tilde{\mathcal{C}}_{X/Y}^0$ of the form*

$$(21) \quad (\langle \mathcal{O}_{\hat{C}} \rangle, \mathcal{O}_{\hat{C}}^{\mathcal{C}, \perp}),$$

where $\mathcal{O}_{\hat{C}}^{\mathcal{C}, \perp}$ is the right orthogonal complement of $\mathcal{O}_{\hat{C}}$ in $\tilde{\mathcal{C}}_{X/Y}^0$.¹

Proof. By Lemma 3.1 and the assumption of the induction, the abelian category $\tilde{\mathcal{C}}_{X/Y}^0$ is the extension closure of some finite number of objects. In particular it is a noetherian abelian category. Hence it is enough to check that $\langle \mathcal{O}_{\hat{C}} \rangle$ is closed under quotients. (cf. [20, Lemma 2.15 (i)].) To prove the latter statement, note that $\langle \mathcal{O}_{\hat{C}} \rangle$ is closed under subobjects since it is a free part of some torsion pair by Lemma 3.1. Also since the self extension of $\mathcal{O}_{\hat{C}}$ vanishes, any object in $\langle \mathcal{O}_{\hat{C}} \rangle$ is a direct sum of $\mathcal{O}_{\hat{C}}$. Let us take an exact sequence in $\tilde{\mathcal{C}}_{X/Y}^0$,

$$0 \rightarrow F \rightarrow \mathcal{O}_{\hat{C}}^{\oplus m} \rightarrow G \rightarrow 0.$$

By the argument above, F is isomorphic to $\mathcal{O}_{\hat{C}}^{\oplus l}$ for some l . Then the object G must be isomorphic to $\mathcal{O}_{\hat{C}}^{\oplus m-l}$, proving that $\langle \mathcal{O}_{\hat{C}} \rangle$ is closed under quotients. \square

By taking the tilting with respect to the torsion pair (21), we define the heart of a bounded t-structure $\mathcal{C}_{X/Y}^0$ on $\mathcal{C}_{X/Y}$ to be,

$$(22) \quad \mathcal{C}_{X/Y}^0 := \langle \mathcal{O}_{\hat{C}}^{\mathcal{C}, \perp}, \mathcal{O}_{\hat{C}}[-1] \rangle.$$

Lemma 3.3. *We have*

$$(23) \quad \mathcal{C}_{X/Y}^0 = \langle S_1, \dots, S_{N-1}, S_N \rangle,$$

where $S_N = \mathcal{O}_{\hat{C}}[-1]$ and S_i for $1 \leq i \leq N-1$ is given by the universal extension in $\tilde{\mathcal{C}}_{X/Y}^0$

$$(24) \quad 0 \rightarrow S'_i \rightarrow S_i \rightarrow \mathcal{O}_{\hat{C}} \otimes \text{Ext}^1(\mathcal{O}_{\hat{C}}, S'_i) \rightarrow 0.$$

¹We put ' \mathcal{C}' ' in the notation of the right orthogonal complement, in order to distinguish it with a similar orthogonal complement in other abelian category in Subsection 4.4.

Proof. We first note the vanishing

$$(25) \quad \mathrm{Hom}(\mathcal{O}_{\widehat{C}}, \mathcal{C}_{X/Y'}^0) = 0,$$

since $\mathbf{R}g_*\mathcal{C}_{X/Y'}^0 \subset \mathrm{Coh}_0(Y')[-1]$. By the vanishing (25), we have $\mathrm{Hom}(\mathcal{O}_{\widehat{C}}, S'_i) = 0$ for $1 \leq i \leq N-1$. Combined with that (24) is the universal extension, it follows that $\mathrm{Hom}(\mathcal{O}_{\widehat{C}}, S_i) = 0$, i.e. $S_i \in \mathcal{O}_{\widehat{C}}^{\mathcal{C}, \perp}$ for $1 \leq i \leq N-1$. Therefore the RHS of (23) is contained in the LHS of (23).

Conversely, let us take an object $E \in \mathcal{O}_{\widehat{C}}^{\mathcal{C}, \perp}$. By Lemma 3.1, there is an exact sequence in $\widetilde{\mathcal{C}}_{X/Y}^0$,

$$0 \rightarrow F \rightarrow E \rightarrow \mathcal{O}_{\widehat{C}} \otimes V \rightarrow 0$$

for some $F \in \mathcal{C}_{X/Y'}^0$ and some finite dimensional \mathbb{C} -vector space V . Since $\mathrm{Hom}(\mathcal{O}_{\widehat{C}}, E) = 0$, we have the injection

$$V \hookrightarrow \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}}, F).$$

Let W be the cokernel of the above injection. There is an exact sequence in $\mathcal{C}_{X/Y}^0$,

$$0 \rightarrow E \rightarrow \widehat{F} \rightarrow \mathcal{O}_{\widehat{C}} \otimes W \rightarrow 0,$$

where \widehat{F} is the universal extension in $\widetilde{\mathcal{C}}_{X/Y}^0$,

$$(26) \quad 0 \rightarrow F \rightarrow \widehat{F} \rightarrow \mathcal{O}_{\widehat{C}} \otimes \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}}, F) \rightarrow 0.$$

It is enough to show that \widehat{F} is contained in the RHS of (23). Since $\mathcal{C}_{X/Y'}^0$ is the extension closure of S'_1, \dots, S'_{N-1} , this follows from the following claim: for an exact sequence in $\mathcal{C}_{X/Y'}^0$,

$$(27) \quad 0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0,$$

suppose that their universal extensions \widehat{F}_i in $\widetilde{\mathcal{C}}_{X/Y}^0$

$$0 \rightarrow F_i \rightarrow \widehat{F}_i \rightarrow \mathcal{O}_{\widehat{C}} \otimes \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}}, F_i) \rightarrow 0$$

are contained in the RHS of (23). Then \widehat{F} is contained in the RHS of (23). To prove this claim, first note that $\mathrm{Hom}(\mathcal{O}_{\widehat{C}}, F_2) = 0$ by the vanishing (25). Therefore applying $\mathrm{Hom}(\mathcal{O}_{\widehat{C}}, *)$ to the sequence (27), we obtain the exact sequence

$$0 \rightarrow \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}}, F_1) \rightarrow \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}}, F) \xrightarrow{\psi} \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}}, F_2).$$

It follows that there is an exact sequence in $\mathcal{C}_{X/Y}^0$,

$$(28) \quad 0 \rightarrow \widehat{F}_1 \rightarrow \widehat{F} \rightarrow \overline{F}_2 \rightarrow 0,$$

where \overline{F}_2 fits into the exact sequence in $\widetilde{\mathcal{C}}_{X/Y}^0$,

$$0 \rightarrow F_2 \rightarrow \overline{F}_2 \rightarrow \mathcal{O}_{\widehat{C}} \otimes \mathrm{Im} \psi \rightarrow 0.$$

We have the exact sequence in $\mathcal{C}_{X/Y}^0$,

$$0 \rightarrow \mathcal{O}_{\widehat{C}} \otimes \text{Cok}(\psi)[-1] \rightarrow \overline{F}_2 \rightarrow \widehat{F}_2 \rightarrow 0.$$

Since \widehat{F}_2 is contained in the RHS of (23), the object \overline{F}_2 is also contained in the RHS of (23). Combined with that \widehat{F}_1 is contained in the RHS of (23), the exact sequence (28) implies that the object \widehat{F} is also contained in the RHS of (23). \square

Moreover we have the following lemma:

Lemma 3.4. *For the objects S_i in Lemma 3.3, we have*

$$(29) \quad \mathbf{R}f_* S_i[1] \in \text{Coh}_0(Y), \quad 1 \leq i \leq N.$$

Proof. The claim for $i = N$ is obvious. Suppose that $1 \leq i \leq N - 1$. Applying $\mathbf{R}g_*$ to the sequence (24), we obtain the distinguished triangle

$$\mathbf{R}g_* S'_i \rightarrow \mathbf{R}g_* S_i \rightarrow \mathcal{O}_C^{\oplus m_i},$$

where $m_i = \dim \text{Ext}^1(\mathcal{O}_{\widehat{C}}, S'_i)$. Since $\mathbf{R}g_* S'_i \cong Q_i[-1]$ for some zero dimensional sheaf Q_i on Y' , the object $\mathbf{R}g_* S_i$ is isomorphic to the two term complex

$$(\mathcal{O}_C^{\oplus m_i} \xrightarrow{\phi} Q_i)$$

with $\mathcal{O}_C^{\oplus m_i}$ located in degree zero. It is enough to check $f_* \text{Ker}(\phi) = 0$, which is equivalent to $H^0(C, \text{Ker}(\phi)) = 0$. If $H^0(C, \text{Ker}(\phi))$ is non-zero, then there is a non-zero section $s \in H^0(C, \mathcal{O}_C^{\oplus m_i})$ satisfying $\phi \circ s = 0$. By adjunction, there is non-zero $\widehat{s} \in H^0(\widehat{C}, \mathcal{O}_{\widehat{C}}^{\oplus m_i})$ such that the composition

$$\mathcal{O}_{\widehat{C}} \xrightarrow{\widehat{s}} \mathcal{O}_{\widehat{C}}^{\oplus m_i} \rightarrow S'_i[1]$$

is zero. Here the right morphism is induced by the extension (24). This contradicts to that (24) is the universal extension. Hence $f_* \text{Ker}(\phi) = 0$, and the condition (29) holds. \square

By the above lemmas, the heart $\mathcal{C}_{X/Y}^0 \subset \mathcal{C}_{X/Y}$ satisfies the desired property (14). As a summary, we have obtained the following proposition:

Proposition 3.5. *Let X be a smooth projective surface. Then for each smooth projective surface Y and a birational morphism $f: X \rightarrow Y$, we can associated the heart of a bounded t -structure $\mathcal{C}_{X/Y}^0 \subset \mathcal{C}_{X/Y}$ satisfying the following conditions:*

- *For any $F \in \mathcal{C}_{X/Y}^0$, the object $\mathbf{R}f_* F[1]$ is a zero dimensional sheaf on Y .*
- *$\mathcal{C}_{X/Y}^0$ is the extension closure of a finite number of objects in $\mathcal{C}_{X/Y}^0$.*

- Suppose that f factors as

$$f: X \xrightarrow{g} Y' \xrightarrow{h} Y,$$

where h is a contraction of a (-1) -curve C on Y' , and $\mathcal{C}_{X/Y'}^0$ is the extension closure of objects S'_1, \dots, S'_{N-1} . Then $\mathcal{C}_{X/Y}^0$ is the extension closure of objects $S_1, \dots, S_{N-1}, S_N := \mathcal{O}_{\widehat{C}}[-1]$, where $\widehat{C} = g^*C$ and S_i is the cone of the universal morphism

$$S_i \rightarrow \mathcal{O}_{\widehat{C}} \otimes \text{Ext}^1(\mathcal{O}_{\widehat{C}}, S'_i) \rightarrow S'_i[1].$$

3.2. Generators of the heart $\mathcal{C}_{X/Y}^0$. In this subsection, we give an explicit description of the generator of $\mathcal{C}_{X/Y}^0$. The description here is not canonical, but useful in constructing stability conditions.

For a birational morphism $f: X \rightarrow Y$ as in the previous subsection, we factorize it into a composition of contractions of (-1) -curves,

$$(30) \quad X = X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \dots \xrightarrow{g_{N-1}} X_N \xrightarrow{g_N} X_{N+1} = Y.$$

The birational morphism

$$g_i: X_i \rightarrow X_{i+1}$$

contracts a single (-1) -curve $C_i \subset X_i$ to a point $p_i \in X_{i+1}$. We also set

$$\begin{aligned} g_{i,j} &:= g_{j-1} \circ \dots \circ g_i: X_i \rightarrow X_j, \\ f_i &:= g_{1,i}: X \rightarrow X_i, \end{aligned}$$

and $\widehat{C}_i := f_i^*C_i$. For $j > i$, we also write $g_{i,j}(C_i)$ as $p_i \in X_j$ by abuse of notation. The curves C_i are classified into two types:

- Type I: for any $j > i$, we have $p_i \notin C_j$.
- Type II: there is $j > i$ so that $p_i \in C_j$. In this case, we define $\kappa(i) > i$ to be the smallest $j > i$ satisfying $p_i \in C_j$.

If C_i is type I, we set $S_i = \mathcal{O}_{\widehat{C}_i}[-1]$. If C_i is type II, we consider the exact sequence of sheaves on X_i

$$(31) \quad 0 \rightarrow \overline{S}_i \rightarrow g_{i,\kappa(i)}^* \mathcal{O}_{C_{\kappa(i)}} \rightarrow \mathcal{O}_{C_i} \rightarrow 0,$$

and set $S_i = \mathbf{L}f_i^* \overline{S}_i (= f_i^* \overline{S}_i)$. The sheaf \overline{S}_i is written as

$$\overline{S}_i = \mathcal{O}_{\overline{g_{i,\kappa(i)}^* C_{\kappa(i)} - C_i}}(-p_i),$$

for $p_i \in C_{\kappa(i)}$.

Proposition 3.6. *In the above notation, we have*

$$\mathcal{C}_{X/Y}^0 = \langle S_1, \dots, S_N \rangle.$$

Proof. We show the proposition by the induction on N . Suppose that the claim holds for $f_N: X \rightarrow X_N$. Then we have

$$\mathcal{C}_{X/X_N}^0 = \langle S'_1, \dots, S'_{N-1} \rangle$$

where S'_i are the objects defined similarly to S_i , applied for the composition

$$X = X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{N-1}} X_N.$$

Noting Proposition 3.5 and $S_N = \mathcal{O}_{\widehat{C}_N}[-1]$, it is enough to show that there is a distinguished triangle for each $1 \leq i \leq N-1$,

$$(32) \quad S_i \rightarrow \mathcal{O}_{\widehat{C}_N} \otimes \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}_N}, S'_i) \rightarrow S'_i[1].$$

For $1 \leq i \leq N-1$, we have the following three cases:

Case 1. C_i is type I for both of $X \rightarrow X_N$ and $X \rightarrow Y$.

In this case, we have $S'_i = S_i = \mathcal{O}_{\widehat{C}_i}[-1]$. Also we have

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}_N}, S'_i) &= \mathrm{Hom}(\mathcal{O}_{\widehat{C}_N}, \mathcal{O}_{\widehat{C}_i}) \\ &= \mathrm{Hom}_{X_N}(\mathcal{O}_{C_N}, \mathcal{O}_{p_i}) \\ &\cong 0 \end{aligned}$$

since $p_i \notin C_N$. Therefore we have the distinguished triangle (32).

Case 2. C_i is type I for $X \rightarrow X_N$ and type II for $X \rightarrow Y$.

In this case, we have $S'_i = \mathcal{O}_{\widehat{C}_i}[-1]$, $\kappa(i) = N$ and $S_i = \mathbf{L}f_i^* \overline{S}_i$. We have

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}_N}, S'_i) &\cong \mathrm{Hom}_{X_N}(\mathcal{O}_{C_N}, \mathcal{O}_{p_i}) \\ &\cong \mathbb{C} \end{aligned}$$

since $p_i \in C_N$. By pulling back the exact sequence (31) to X via f_i , we have the distinguished triangle (32).

Case 3. C_i is type II for both of $X \rightarrow X_N$ and $X \rightarrow Y$.

In this case, $1 \leq \kappa(i) \leq N-1$ and $S'_i = S_i = \mathbf{L}f_i^* \overline{S}_i$. We have

$$(33) \quad \begin{aligned} \mathrm{Ext}^1(\mathcal{O}_{\widehat{C}_N}, S'_i) &\cong \mathrm{Ext}_{X_i}^1(\mathbf{L}g_{i,N}^* \mathcal{O}_{C_N}, \overline{S}_i) \\ &\cong \mathrm{Ext}_{X_N}^1(\mathcal{O}_{C_N}, \mathbf{R}g_{i,N*} \overline{S}_i). \end{aligned}$$

Applying $\mathbf{R}g_{i,\kappa(i)*}$ to the sequence (31), we obtain the distinguished triangle

$$\mathbf{R}g_{i,\kappa(i)*} \overline{S}_i \rightarrow \mathcal{O}_{C_{\kappa(i)}} \rightarrow \mathcal{O}_{p_i},$$

such that the right morphism is non-trivial since $p_i \in C_{\kappa(i)}$. Therefore we have $\mathbf{R}g_{i,\kappa(i)*} \overline{S}_i \cong \mathcal{O}_{C_{\kappa(i)}}(-1)$ and

$$\begin{aligned} \mathbf{R}g_{i,N*} \overline{S}_i &\cong \mathbf{R}g_{\kappa(i),N*} \mathcal{O}_{C_{\kappa(i)}}(-1) \\ &\cong 0. \end{aligned}$$

Therefore (33) vanishes and we have the distinguished triangle (32). \square

Here we give some examples.

Example 3.7. (i) Suppose that $f: X \rightarrow Y$ contracts disjoint (-1) -curves C_1, \dots, C_N on X , i.e. all the curves C_i are type I. Then we have

$$\mathcal{C}_{X/Y}^0 = \langle \mathcal{O}_{C_1}[-1], \dots, \mathcal{O}_{C_N}[-1] \rangle.$$

(ii) Suppose that $N = 2$ with $p_1 \in C_2$. Then we have

$$\mathcal{C}_{X/Y}^0 = \langle \mathcal{O}_{\overline{C}_2}(-1), \mathcal{O}_{C_1+\overline{C}_2}[-1] \rangle.$$

Here $\overline{C}_i \subset X$ is the strict transform.

(iii) Suppose that $N = 3$, C_1, C_2 are type II with $\kappa(1) = \kappa(2) = 3$. Then we have

$$\mathcal{C}_{X/Y}^0 = \langle \mathcal{O}_{C_2+\overline{C}_3}(-p_1), \mathcal{O}_{C_1+\overline{C}_3}(-p_2), \mathcal{O}_{C_1+C_2+\overline{C}_3}[-1] \rangle.$$

Note that, by [6, Lemma 3.1], the category $\mathcal{C}_{X/Y} \cap \text{Coh}(X)$ is shown to be the heart of a bounded t-structure on $\mathcal{C}_{X/Y}$. As the above example indicates, the heart $\mathcal{C}_{X/Y}^0$ can be shown to be the tilting with respect to a certain torsion pair on $\mathcal{C}_{X/Y} \cap \text{Coh}(X)$. We have the following:

Corollary 3.8. Let $f: X \rightarrow Y$ be a birational morphism between smooth projective surfaces, and $Y' \rightarrow Y$ the blow up at $f(\text{Ex}(f)) \subset Y$. Let $C_1, \dots, C_l \subset Y'$ be the exceptional curves of $Y' \rightarrow Y$, and $\widehat{C}_i \subset X$ the proper pull-back of C_i to X . By setting $\mathcal{S} := \langle \mathcal{O}_{\widehat{C}_1}, \dots, \mathcal{O}_{\widehat{C}_l} \rangle$ in $\mathcal{C}_{X/Y} \cap \text{Coh}(X)$, we have

$$(34) \quad \mathcal{C}_{X/Y}^0 = \langle \mathcal{S}^\perp, \mathcal{S}[-1] \rangle.$$

Proof. Since we will not use this result, we make the proof brief. A proof similar to Lemma 3.2 shows that there is a torsion pair on $\mathcal{C}_{X/Y} \cap \text{Coh}(X)$ of the form $(\mathcal{S}, \mathcal{S}^\perp)$. Hence the RHS is the heart of a bounded t-structure on $\mathcal{C}_{X/Y}$. Also any type I curve appears as an exceptional curve of a blow-up at a point in Y . Therefore it is straightforward to check that all the objects S_i in Proposition 3.6 are contained in the RHS of (34). Since both sides of (34) are hearts of bounded t-structures on $\mathcal{C}_{X/Y}$, the equality (34) holds. \square

3.3. t-structures on $D^b \text{Coh}(X)$. Let $f: X \rightarrow Y$ be a birational morphism as in the previous subsections. Let

$$\mathcal{A}_Y \subset D^b \text{Coh}(Y)$$

be the heart of a bounded t-structure such that $\mathcal{O}_y \in \mathcal{A}_Y$ for any $y \in Y$. We construct the heart of a t-structure on $D^b \text{Coh}(X)$ by gluing \mathcal{A} and $\mathcal{C}_{X/Y}^0$ constructed in the previous subsections.

Let us consider the following sequence of exact functors

$$\mathcal{C}_{X/Y} \rightarrow D^b \text{Coh}(X) \xrightarrow{\mathbf{R}f^!} D^b \text{Coh}(Y),$$

where the left functor is the natural inclusion. It is straightforward to check that the above sequence is an exact triple as in Subsection 2.2. By gluing \mathcal{A}_Y and $\mathcal{C}_{X/Y}^0$, we obtain the heart

$$\mathcal{A}_X \subset D^b \text{Coh}(X).$$

The heart \mathcal{A}_X is described as follows:

Lemma 3.9. *We have*

$$(35) \quad \mathcal{A}_X = \langle \mathcal{C}_{X/Y}^0, \mathbf{L}f^* \mathcal{A}_Y \rangle,$$

and $(\mathcal{C}_{X/Y}^0, \mathbf{L}f^* \mathcal{A}_Y)$ is a torsion pair on \mathcal{A}_X .

Proof. The proof is very similar to Lemma 3.1. First we show that the RHS of (35) is contained in the LHS of (35). It is obvious that $\mathcal{C}_{X/Y}^0$ is contained in the LHS, so we show that $\mathbf{L}f^* \mathcal{A}_Y$ is contained in the LHS. For $M \in \mathcal{A}_Y$, we have $\mathbf{R}f_! \mathbf{L}f^* M \cong M \in \mathcal{A}_Y$ and

$$(36) \quad \text{Hom}(\mathcal{C}_{X/Y}, \mathbf{L}f^* M) \cong 0$$

by the adjunction. Also we have

$$(37) \quad \begin{aligned} \text{Hom}(\mathbf{L}f^* M, \mathcal{C}_{X/Y}^{\geq 0}) &\cong \text{Hom}(M, \mathbf{R}f_* \mathcal{C}_{X/Y}^{\geq 0}) \\ &\cong 0 \end{aligned}$$

since $\mathbf{R}f_* \mathcal{C}_{X/Y}^{\geq 0} \subset D^{>0} \text{Coh}(Y)$ with cohomology sheaves zero dimensional, and $\text{Coh}_0(Y) \subset \mathcal{A}_Y$. Therefore $\mathbf{L}f^* M$ is an object in \mathcal{A}_X by the definition of the gluing.

Conversely, we show that \mathcal{A}_X is contained in the RHS of (35). For an object $E \in \mathcal{A}_X$, there is a distinguished triangle

$$F \rightarrow E \rightarrow \mathbf{L}f^* \mathbf{R}f_! E,$$

with $F \in \mathcal{C}_{X/Y}$. Similarly to the proof of Lemma 3.1, we have the exact sequence in \mathcal{A}_X ,

$$0 \rightarrow \mathcal{H}_{\mathcal{C}_{X/Y}^0}^0(F) \rightarrow E \rightarrow \mathbf{L}f^* \mathbf{R}f_! E \rightarrow \mathcal{H}_{\mathcal{C}_{X/Y}^0}^1(F) \rightarrow 0$$

and $\mathcal{H}_{\mathcal{C}_{X/Y}^0}^i(F) = 0$ for $i \neq 0, 1$. By the vanishing (37), we also have $\mathcal{H}_{\mathcal{C}_{X/Y}^0}^1(F) = 0$ and $F \in \mathcal{C}_{X/Y}^0$. Consequently we have the exact sequence in \mathcal{A}_X

$$(38) \quad 0 \rightarrow F \rightarrow E \rightarrow \mathbf{L}f^* \mathbf{R}f_! E \rightarrow 0$$

with $F \in \mathcal{C}_{X/Y}^0$. Therefore E is contained in the LHS of (35). By (36) and (38), the pair $(\mathcal{C}_{X/Y}^0, \mathbf{L}f^* \mathcal{A}_Y)$ is a torsion pair on \mathcal{A}_X . \square

4. PROOF OF THEOREM 1.2

In this section, we construct a connected open subset

$$U(Y) \subset \text{Stab}(X)_{\mathbb{R}}$$

for each birational morphism $f: X \rightarrow Y$, and prove Theorem 1.2. In what follows, we always assume that $f: X \rightarrow Y$ is a birational morphism between smooth projective surfaces.

4.1. Central charges corresponding to $U(Y)$. Let

$$\text{NS}_f(X)_{\mathbb{R}} \subset \text{NS}(X)_{\mathbb{R}}$$

be the orthogonal complement of $f^*\text{NS}(Y)$ with respect to the intersection pairing. Note that $\text{NS}_f(X)_{\mathbb{R}}$ is a linear subspace of $\text{NS}(X)_{\mathbb{R}}$ spanned by the irreducible components of the exceptional locus of f . For fixed $k > 0$, we set

$$C_{f,k}(X) := \left\{ D \in \text{NS}_f(X)_{\mathbb{R}} : \begin{array}{l} D \cdot c_1(F) > 0 \text{ for all} \\ F \in \mathcal{C}_{X/Y}^0, \quad D^2 + k > 0. \end{array} \right\}.$$

We have the following lemma:

Lemma 4.1. *$C_{f,k}(X)$ is a non-empty connected open subset of $\text{NS}_f(X)_{\mathbb{R}}$.*

Proof. We factorize $f: X \rightarrow Y$ into the composition of blow-downs as in (30). In the notation of Subsection 3.2, we have

$$\text{NS}_f(X)_{\mathbb{R}} = \bigoplus_{i=1}^N \mathbb{R}[\widehat{C}_i]$$

for $\widehat{C}_i = f_i^* C_i$. For $D \in \text{NS}_f(X)_{\mathbb{R}}$, it is contained in $C_{f,k}(X)$ if and only if $D \cdot c_1(S_i) > 0$ for all $1 \leq i \leq N$, where S_i is given in Subsection 3.2, and $D^2 + k > 0$. If we write $D \in \text{NS}_f(X)_{\mathbb{R}}$ as

$$D = \sum_{i=1}^N t_i [\widehat{C}_i]$$

for $t_i \in \mathbb{R}$, then $D \cdot c_1(S_i)$ is calculated as

$$D \cdot c_1(S_i) = \begin{cases} t_i, & i \text{ is type I,} \\ t_i - t_{\kappa(i)}, & i \text{ is type II.} \end{cases}$$

Therefore $C_{f,k}(X)$ is identified with

$$C_{f,k}(X) = \left\{ (t_1, \dots, t_N) \in \mathbb{R}^N : \begin{array}{l} t_i > 0, \quad i \text{ is type I,} \\ t_i > t_{\kappa(i)}, \quad i \text{ is type II,} \\ t_1^2 + \dots + t_N^2 < k. \end{array} \right\}.$$

Hence $C_{f,k}(X)$ is a non-empty connected open subset of $\text{NS}_f(X)_{\mathbb{R}}$. \square

We consider the following sets

$$\begin{aligned} A^\dagger(Y) &:= \{f^*\omega + D : \omega \in A(Y), D \in C_{f,\omega^2}(X)\}, \\ \overline{A}^\dagger(Y) &:= \{f^*\omega + D : \omega \in \overline{A}(Y), D \in C_{f,\omega^2}(X)\}. \end{aligned}$$

The set $A^\dagger(Y)$ is a topological fiber bundle

$$f_* : A^\dagger(Y) \rightarrow A(Y)$$

with fiber at ω is $C_{f,\omega^2}(X)$. By Lemma 4.1, $A^\dagger(Y)$ is an open connected subset of $\text{NS}(X)_\mathbb{R}$, and $\overline{A}^\dagger(Y)$ is its partial compactification. We will consider the central charges of the form

$$Z_{f^*\omega+D} \in N(X)_\mathbb{C}^\vee, \quad f^*\omega + D \in \overline{A}^\dagger(Y).$$

The compatible t-structure will be given in the next subsection.

4.2. t-structures corresponding to $U(Y)$. For a rational point $\omega \in \overline{A}(Y)$, we have the heart of a bounded t-structure

$$\mathcal{A}_\omega \subset D^b \text{Coh}(Y)$$

constructed in Subsection 2.4. By the construction, all the objects \mathcal{O}_y for $y \in Y$ are contained in \mathcal{A}_ω . Therefore Lemma 3.9 implies the existence of a bounded t-structure on $D^b \text{Coh}(X)$ with heart given by

$$(39) \quad \mathcal{A}_\omega(X/Y) := \langle \mathcal{C}_{X/Y}^0, \mathbf{L}f^*\mathcal{A}_\omega \rangle,$$

such that $(\mathcal{C}_{X/Y}^0, \mathbf{L}f^*\mathcal{A}_\omega)$ is a torsion pair on $\mathcal{A}_\omega(X/Y)$. Later we will need the following lemma:

Lemma 4.2. *The subcategories*

$$\mathcal{C}_{X/Y}^0, \mathbf{L}f^*\mathcal{A}_\omega \subset \mathcal{A}_\omega(X/Y)$$

are closed under subobjects and quotients.

Proof. Since $(\mathcal{C}_{X/Y}^0, \mathbf{L}f^*\mathcal{A}_\omega)$ is a torsion pair on $\mathcal{A}_\omega(X/Y)$, the subcategory $\mathcal{C}_{X/Y}^0$ is closed under quotients, and the subcategory $\mathbf{L}f^*\mathcal{A}_\omega$ is closed under subobjects. For $F \in \mathcal{C}_{X/Y}^0$, suppose that $A \hookrightarrow F$ is an injection in $\mathcal{A}_\omega(X/Y)$. Then it induces an injection $\mathbf{R}f_!A \hookrightarrow \mathbf{R}f_!F$ in \mathcal{A}_ω . Since $\mathbf{R}f_!F = 0$, we have $\mathbf{R}f_!A = 0$, hence $A \in \mathcal{C}_{X/Y}^0$. This implies that $\mathcal{C}_{X/Y}^0$ is also closed under subobjects.

For $M \in \mathcal{A}_\omega$, let us take an exact sequence in $\mathcal{A}_\omega(X/Y)$

$$0 \rightarrow E_1 \rightarrow \mathbf{L}f^*M \rightarrow E_2 \rightarrow 0.$$

By the argument as above, we have $E_1 \in \mathbf{L}f^*\mathcal{A}_\omega$. For any $F \in \mathcal{C}_{X/Y}^0$, we have

$$\mathbf{R} \text{Hom}(F, \mathbf{L}f^*\mathcal{A}_\omega) = 0$$

since $\mathbf{R}f_!F = 0$. Therefore we have $\text{Hom}(F, E_2) = 0$, hence $E_2 \in \mathbf{L}f^*\mathcal{A}_\omega$ since $(\mathcal{C}_{X/Y}^0, \mathbf{L}f^*\mathcal{A}_\omega)$ is a torsion pair on $\mathcal{A}_\omega(X/Y)$. This implies that $\mathbf{L}f^*\mathcal{A}_\omega$ is also closed under quotients. \square

We will also need the following lemma:

Lemma 4.3. *The abelian category $\mathcal{A}_\omega(X/Y)$ is noetherian.*

Proof. Suppose that there is an infinite sequence of surjections in $\mathcal{A}_\omega(X/Y)$

$$(40) \quad E = E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_i \twoheadrightarrow E_{i+1} \twoheadrightarrow \cdots .$$

Applying $\mathbf{R}f_!$ to the sequence (40), we obtain surjections

$$(41) \quad \mathbf{R}f_! E_i \twoheadrightarrow \mathbf{R}f_! E_{i+1}$$

in $\mathcal{A}_\omega \subset D^b \text{Coh}(Y)$. Since \mathcal{A}_ω is noetherian by the proof of [21, Lemma 5.2], we may assume that (41) are isomorphism for all i . Hence if we take the exact sequences in $\mathcal{A}_\omega(X/Y)$

$$0 \rightarrow F_i \rightarrow E \rightarrow E_i \rightarrow 0,$$

then $F_i \in \mathcal{C}_{X/Y}^0$. On the other hand, we have the exact sequence in $\mathcal{A}_\omega(X/Y)$

$$0 \rightarrow F \rightarrow E \rightarrow \mathbf{L}f^* M \rightarrow 0$$

for $F \in \mathcal{C}_{X/Y}^0$ and $M \in \mathcal{A}_\omega$. Since $\text{Hom}(F_i, \mathbf{L}f^* M) = 0$, we have the sequence of injections in $\mathcal{A}_\omega(X/Y)$

$$F_1 \hookrightarrow F_2 \hookrightarrow \cdots \hookrightarrow F.$$

By Lemma 4.2, the above sequence is a sequence of injections in $\mathcal{C}_{X/Y}^0$. Since $\mathcal{C}_{X/Y}^0$ is the extension closure of a finite number of objects, it is noetherian, hence the above sequence terminates. Therefore the sequence (40) also terminates. \square

4.3. Construction of $U(Y)$. For $f^*\omega + D \in \overline{A}^\dagger(Y)$ with ω, D rational, we consider the pair

$$(42) \quad \sigma_{f^*\omega + D} := (Z_{f^*\omega + D}, \mathcal{A}_\omega(X/Y)).$$

The purpose here is to show that $\sigma_{f^*\omega + D}$ gives a point in $\text{Stab}(X)_\mathbb{R}$. We first prepare a lemma: let us consider the central charge $Z_{\omega, D} \in N(Y)_\mathbb{C}^\vee$ defined by

$$Z_{\omega, D}(M) := Z_\omega(M) + \frac{D^2}{2} \text{ch}_0(M).$$

Note that, for any $M \in D^b \text{Coh}(Y)$, it satisfies the following equality:

$$(43) \quad Z_{f^*\omega + D}(\mathbf{L}f^* M) = Z_{\omega, D}(M).$$

Lemma 4.4. *We have*

$$\sigma_{\omega, D} := (Z_{\omega, D}, \mathcal{A}_\omega) \in \text{Stab}(Y).$$

Proof. Note that the pair $(Z_\omega, \mathcal{A}_\omega)$ is shown to be an element of $\text{Stab}(Y)$ in [21, Lemma 3.12], and almost the same argument is applied. Indeed for a non-zero $M \in \mathcal{A}_\omega$, $Z_{\omega,D}(M)$ is written as

$$-\text{ch}_2(M) + \frac{\text{ch}_0(M)}{2}(D^2 + \omega^2) + i \text{ch}_1(M) \cdot \omega.$$

By the construction of \mathcal{A}_ω , we have $\text{ch}_1(M) \cdot \omega \geq 0$. Moreover, the proof of [21, Lemma 3.12] shows that if $\text{ch}_1(M) \cdot \omega = 0$, then M satisfies either $\text{ch}_0(M) < 0, \text{ch}_2(M) \geq 0$ or $\text{ch}_0(M) = 0, \text{ch}_2(M) > 0$. By the definition of $C_{f,\omega^2}(X)$, we have $D^2 + \omega^2 > 0$, hence $Z_{\omega,D}$ satisfies the property (6).

The proofs for other properties are also the same as in [21, Lemma 3.12]. Indeed the abelian category \mathcal{A}_ω is noetherian, so there is no need to modify the proof for the Harder-Narasimhan property. As for the support property, since $D \in C_{f,s^2\omega^2}(X)$ for any $s \geq 1$, the wall-crossing method in [21, Theorem 3.23] for the family $\{\sigma_{s\omega,D}\}_{s \geq 1}$ works as well. This implies that the Chern characters of $Z_{\omega,D}$ -semistable objects in \mathcal{A}_ω satisfies the same Bogomolov-Gieseker type inequality as in [21, Theorem 3.23], and the same computation in the proof of [21, Lemma 3.12] shows the support property for $\sigma_{\omega,D}$. Since there is no need to modify the proof, we omit the detail. \square

Using the above lemma, we show the following:

Lemma 4.5. *In the above situation, the pair (4.2) is a stability condition on $D^b \text{Coh}(X)$.*

Proof. We first check that $\sigma_{f^*\omega+D}$ satisfies the property (6). For non-zero $F \in \mathcal{C}_{X/Y}^0$ and $M \in \mathcal{A}_\omega$, we have the equality (43) and

$$(44) \quad \text{Im } Z_{f^*\omega+D}(F) = c_1(F) \cdot D > 0.$$

Combined with Lemma 4.4 and the fact that $\mathcal{A}_\omega(X/Y)$ is the extension closure of $\mathcal{C}_{X/Y}^0$ and $\mathbf{L}f^*\mathcal{A}_\omega$, it follows that $\sigma_{f^*\omega+D}$ satisfies the property (6).

In order to show the Harder-Narasimhan property, since $\mathcal{A}_\omega(X/Y)$ is noetherian by Lemma 4.3, it is enough to show that there is no infinite sequence

$$(45) \quad E = E_1 \supset E_2 \supset \cdots \supset E_i \supset E_{i+1} \supset \cdots,$$

in $\mathcal{A}_\omega(X/Y)$ such that

$$(46) \quad \arg Z_{f^*\omega+D}(E_{i+1}) > \arg Z_{f^*\omega+D}(E_i/E_{i+1})$$

for all i . (cf. [7, Proposition 2.4].) Suppose that a sequence (45) satisfying (46) exists. Since $\text{Im } Z_{f^*\omega+D}(*)$ is discrete by the rationality of ω and D , we may assume that $\text{Im } Z_{f^*\omega+D}(E_i)$ is constant, hence $\text{Im } Z_{f^*\omega+D}(E_i/E_{i+1}) = 0$. This implies that $\arg Z_{f^*\omega+D}(E_i/E_{i+1}) = \pi$, which contradicts to (46). \square

We also have the following lemma:

Lemma 4.6. *An object $M \in \mathcal{A}_\omega$ is $Z_{\omega,D}$ -(semi)stable if and only if $\mathbf{L}f^*M \in \mathcal{A}_\omega(X/Y)$ is $Z_{f^*\omega+D}$ -(semi)stable.*

Proof. Since the equality (43) holds, the lemma obviously follows from Lemma 4.2. \square

The final step is to show the support property for the pair (42). We have the following proposition:

Proposition 4.7. *In the above situation, we have*

$$\sigma_{f^*\omega+D} \in \text{Stab}(X)_{\mathbb{R}},$$

i.e. $\sigma_{f^\omega+D}$ satisfies the support property.*

Proof. We first note that, for any $F \in \mathcal{C}_{X/Y}^0$, we have

$$Z_{f^*\omega+D}(F) = Z_D(F),$$

and its imaginary part is positive by (44). Since $\mathcal{C}_{X/Y}^0$ is the extension closure of a finite number of objects, there is $0 < \theta \leq 1$ so that

$$(47) \quad Z_D(\mathcal{C}_{X/Y}^0 \setminus \{0\}) \subset \mathbb{H}_\theta,$$

where \mathbb{H}_θ is defined by

$$\mathbb{H}_\theta := \{r \exp(i\pi\phi) : r > 0, \phi \in [\theta, 1]\}.$$

We can find a constant $K(\theta) > 0$, which only depends on θ , satisfying the following: for any $k \geq 1$ and $z_1, \dots, z_k \in \mathbb{H}_\theta$, we have

$$(48) \quad \frac{|z_1 + \dots + z_k|}{|z_1| + \dots + |z_k|} \geq K(\theta).$$

For instance, one can take $K(\theta) = \sin^2 \pi\theta/2$. The proof of this fact is an easy exercise, and we omit the proof.

Let us take a $Z_{f^*\omega+D}$ -semistable object $E \in \mathcal{A}_\omega(X/Y)$. We have the exact sequence in $\mathcal{A}_\omega(X/Y)$

$$(49) \quad 0 \rightarrow F \rightarrow E \rightarrow \mathbf{L}f^*M \rightarrow 0$$

for $F \in \mathcal{C}_{X/Y}^0$ and $M \in \mathcal{A}_\omega$. We find a constant K in Definition 2.2 by dividing into the three cases:

Case 1. $M = 0$, i.e. $E \in \mathcal{C}_{X/Y}^0$.

In this case, let us take $K' > 0$ so that the following holds:

$$\frac{\|S_i\|}{|Z_D(S_i)|} < K',$$

for all $1 \leq i \leq N$. Here S_1, \dots, S_N are the objects in $\mathcal{C}_{X/Y}^0$ as in Proposition 3.6. Then by (48), it follows that

$$\frac{\|E\|}{|Z_{f^*\omega+D}(E)|} < \frac{K'}{K(\theta)} (= K).$$

Note that the $Z_{f^*\omega+D}$ -stability of E is not needed in the above argument.

Case 2. $F = 0$, *i.e.* $E \cong \mathbf{L}f^*M$.

In this case, the object M is $Z_{\omega,D}$ -semistable by Lemma 4.6. By Lemma 4.4, the pair $(Z_{\omega,D}, \mathcal{A}_\omega)$ satisfies the support property. Therefore we can find $K > 0$, which is independent of M , so that

$$\frac{\|E\|}{|Z_{f^*\omega+D}(E)|} = \frac{\|M\|}{|Z_{\omega,D}(M)|} < K.$$

Case 3. $F \neq 0$ and $M \neq 0$.

In this case, note that the object M may not be $Z_{\omega,D}$ -semistable. So there may be an exact sequence in \mathcal{A}_ω

$$0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$$

satisfying that

$$(50) \quad \arg Z_{\omega,D}(M'') > \arg Z_{\omega,D}(M) > \arg Z_{\omega,D}(M').$$

We have the surjections in $\mathcal{A}_\omega(X/Y)$

$$E \twoheadrightarrow \mathbf{L}f^*M \twoheadrightarrow \mathbf{L}f^*M',$$

such that the kernel of their composition has the numerical class $[F] + [\mathbf{L}f^*M'']$. By the $Z_{f^*\omega+D}$ -stability of E , we have

$$(51) \quad \begin{aligned} \arg(Z_D(F) + Z_{\omega,D}(M'')) &= \arg Z_{f^*\omega+D}(F \oplus \mathbf{L}f^*M'') \\ &\leq \arg Z_{\omega,D}(M'). \end{aligned}$$

On the other hand, by (47), the exact sequence (49) and the $Z_{f^*\omega+D}$ -stability of E , we have the inequalities

$$(52) \quad \pi\theta \leq \arg Z_D(F) \leq \arg Z_{\omega,D}(M).$$

The inequalities (50), (51) and (52) imply that $\arg Z_{\omega,D}(M') \geq \pi\theta$. Let us take the $Z_{\omega,D}$ -semistable factors of M ,

$$M_1, \dots, M_k \in \mathcal{A}_\omega.$$

Then the above argument implies that $Z_{\omega,D}(M_i) \in \mathbb{H}_\theta$ for all $1 \leq i \leq k$. Let $K > 0$ be a constant which we took in the previous cases. Then we have

$$\begin{aligned} \frac{\|E\|}{|Z_{f^*\omega+D}(E)|} &\leq \frac{1}{K(\theta)} \cdot \frac{\|F\| + \sum_{i=1}^k \|M_i\|}{|Z_D(F)| + \sum_{i=1}^k |Z_{D,\omega}(M_i)|} \\ &\leq \frac{K}{K(\theta)}, \end{aligned}$$

by (48) and the results in the previous steps. Therefore $\sigma_{f^*\omega+D}$ satisfies the support property. \square

For an irrational $f^*\omega + D$, we have the following analogue of Proposition 2.8:

Proposition 4.8. *The embedding $\overline{A}^\dagger(Y) \subset \mathrm{NS}(X)_\mathbb{R}$ lifts to a continuous map*

$$(53) \quad \sigma_Y: \overline{A}^\dagger(Y) \rightarrow \mathrm{Stab}(X)_\mathbb{R}$$

which takes any rational point $f^\omega + D$ in $\overline{A}^\dagger(Y)$ to the stability condition $\sigma_{f^*\omega+D}$ in Proposition 4.7.*

Proof. The proof will be given in Subsection 5.2. \square

We define $U(Y)$ to be

$$U(Y) := \sigma_Y(A^\dagger(Y)) \subset \mathrm{Stab}(X)_\mathbb{R}.$$

Note that $U(Y)$ is a connected open subset of $\mathrm{Stab}(X)_\mathbb{R}$, which is homeomorphic to $A^\dagger(Y)$ under the forgetting map $\mathrm{Stab}(X)_\mathbb{R} \rightarrow \mathrm{NS}(X)_\mathbb{R}$.

4.4. Relations of $U(Y)$ under blow-downs. In the situation of the previous subsections, suppose that f factors as

$$f: X \xrightarrow{g} Y' \xrightarrow{h} Y$$

where h contracts a single (-1) -curve C on Y' to a point in Y . The purpose of this subsection is to prove that $\overline{U}(Y) \cap \overline{U}(Y')$ is non-empty with real codimension one.

We first see the relationship between the hearts of bounded t-structures (39) under blow-downs. Let us take a rational point $\omega \in A(Y)$, and consider $h^*\omega \in \overline{A}(Y')$.

Lemma 4.9. *There is a torsion pair on $\mathcal{A}_{h^*\omega}(X/Y')$ of the form*

$$(54) \quad (\langle \mathcal{O}_{\widehat{C}} \rangle, \mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp})$$

*where $\widehat{C} = g^*C$ and $\mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp}$ is the right orthogonal complement of $\mathcal{O}_{\widehat{C}}$ in $\mathcal{A}_{h^*\omega}(X/Y')$.*

Proof. Since $\mathcal{O}_C \in \mathrm{Per}(Y'/Y)$, we have $\mathcal{O}_{\widehat{C}} \in \mathcal{A}_{h^*\omega}(X/Y')$. Also the abelian category $\mathcal{A}_{h^*\omega}(X/Y')$ is noetherian by Lemma 4.3, so it is enough to check that $\langle \mathcal{O}_{\widehat{C}} \rangle$ is closed under quotients in $\mathcal{A}_{h^*\omega}(X/Y')$. (cf. [20, Lemma 2.15 (i)].) Let us take an exact sequence in $\mathcal{A}_{h^*\omega}(X/Y')$

$$0 \rightarrow E_1 \rightarrow \mathcal{O}_{\widehat{C}}^{\oplus m} \rightarrow E_2 \rightarrow 0$$

for $m \in \mathbb{Z}_{\geq 1}$. By Lemma 4.2, E_i is of the form $\mathbf{L}g^*M_i$ for some $M_i \in \mathcal{A}_{h^*\omega}$, and we have the exact sequence in $\mathcal{A}_{h^*\omega}$

$$0 \rightarrow M_1 \rightarrow \mathcal{O}_{\widehat{C}}^{\oplus m} \rightarrow M_2 \rightarrow 0.$$

Since \mathcal{O}_C is a simple object in $\mathrm{Per}(Y'/Y)$, (cf. [9, Proposition 3.5.8].) it easily follows that $\mathcal{O}_{\widehat{C}}$ is also a simple object in $\mathcal{A}_{h^*\omega}$. Hence $M_i \in \langle \mathcal{O}_{\widehat{C}} \rangle$ and $E_i \in \langle \mathcal{O}_{\widehat{C}} \rangle$ follows. This implies that $\langle \mathcal{O}_{\widehat{C}} \rangle$ is closed under quotients. \square

The abelian categories $\mathcal{A}_\omega(X/Y)$ and $\mathcal{A}_{h^*\omega}(X/Y')$ are related as follows:

Lemma 4.10. *In the above situation, we have*

$$\mathcal{A}_\omega(X/Y) = \langle \mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp}, \mathcal{O}_{\widehat{C}}[-1] \rangle,$$

i.e. $\mathcal{A}_\omega(X/Y)$ is the tilting with respect to the torsion pair (54).

Proof. Since both sides are the hearts of bounded t-structures, it is enough to show that the LHS is contained in the RHS. This is equivalent to the following inclusions:

$$(55) \quad \mathcal{C}_{X/Y}^0 \subset \langle \mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp}, \mathcal{O}_{\widehat{C}}[-1] \rangle,$$

$$(56) \quad \mathbf{L}f^* \mathcal{A}_\omega \subset \langle \mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp}, \mathcal{O}_{\widehat{C}}[-1] \rangle.$$

We first show the inclusion (55). By the construction of $\mathcal{C}_{X/Y}^0$ in (22), it is enough to show

$$(57) \quad \mathcal{O}_{\widehat{C}}^{\mathcal{C}, \perp} \subset \mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp}.$$

Since $\mathcal{O}_{\widehat{C}} \in \mathbf{L}g^* \mathcal{A}_{h^*\omega}$, we have the following inclusion,

$$\langle \mathcal{C}_{X/Y'}^0, \mathcal{O}_{\widehat{C}} \rangle \subset \langle \mathcal{C}_{X/Y'}^0, \mathbf{L}g^* \mathcal{A}_{h^*\omega} \rangle,$$

or equivalently $\widetilde{\mathcal{C}}_{X/Y'}^0 \subset \mathcal{A}_{h^*\omega}(X/Y')$. The inclusion (57) obviously follows from the above inclusion.

Next we show the inclusion (56). By Lemma 2.5, we have the inclusions

$$\mathbf{L}f^* \mathcal{A}_\omega \subset \mathbf{L}g^* \mathcal{A}_{h^*\omega} \subset \mathcal{A}_{h^*\omega}(X/Y').$$

Also we have

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_{\widehat{C}}, \mathbf{L}f^* \mathcal{A}_\omega) &\cong \mathrm{Hom}(\mathbf{R}h_! \mathcal{O}_C, \mathcal{A}_\omega) \\ &\cong 0 \end{aligned}$$

since $\mathbf{R}h_! \mathcal{O}_C = 0$. This implies that $\mathbf{L}f^* \mathcal{A}_\omega$ is contained in $\mathcal{O}_{\widehat{C}}^{\mathcal{A}, \perp}$, proving (56). \square

Note that $h^*A(Y)$ is a real codimension one boundary of $\overline{A}(Y')$. We define the subset $A_h^\dagger(Y) \subset \overline{A}^\dagger(Y')$ by the Cartesian square

$$\begin{array}{ccc} A_h^\dagger(Y) & \longrightarrow & \overline{A}^\dagger(Y') \\ g_* \downarrow & \square & \downarrow g_* \\ h^*A(Y) & \longrightarrow & \overline{A}(Y'). \end{array}$$

By Proposition 4.8, we have

$$(58) \quad \sigma_{Y'}(A_h^\dagger(Y)) \subset \overline{U}(Y')$$

and it is a real codimension one boundary of $\overline{U}(Y')$. The following proposition shows the desired property of $\overline{U}(Y) \cap \overline{U}(Y')$.

Proposition 4.11. *We have*

$$\sigma_{Y'}(A_h^\dagger(Y)) \subset \overline{U}(Y).$$

Proof. It is enough to show the claim for rational points in $A_h^\dagger(Y)$. Let us take a rational point in $A_h^\dagger(Y)$,

$$g^*h^*\omega + D = f^*\omega + D \in A_h^\dagger(Y)$$

for $\omega \in A(Y)$ and $D \in C_{g,\omega^2}(X)$. By (58), we have the point

$$(59) \quad \sigma_{f^*\omega+D} \in \overline{U}(Y').$$

On the other hand, if we take a rational number $0 < t \ll 1$, which is sufficiently small depending on ω and D , we have

$$(60) \quad f^*\omega + D + t\widehat{C} \in A^\dagger(Y),$$

by the description of $C_{f,\omega^2}(X)$ in the proof of Lemma 4.1. Hence we have the point

$$\sigma_{f^*\omega+D+t\widehat{C}} \in U(Y).$$

It is enough to show that

$$(61) \quad \lim_{t \rightarrow +0} \sigma_{f^*\omega+D+t\widehat{C}} = \sigma_{f^*\omega+D}.$$

The relation (61) obviously follows at the level of central charges. Also the hearts of bounded t-structures associated to (59), (60) are

$$\mathcal{A}_{h^*\omega}(X/Y'), \quad \mathcal{A}_\omega(X/Y),$$

respectively. By Lemma 4.10, these t-structures are related by a tilting. Moreover the heart $\mathcal{A}_\omega(X/Y)$ is independent of t . Therefore we can apply Lemma 4.12 below, and conclude that the relation (61) holds. \square

We have used the following lemma, which is proved in [24].

Lemma 4.12. ([24, Lemma 7.1]) *Let $\mathcal{A}, \mathcal{A}'$ be the hearts of bounded t-structures on \mathcal{D} , which are related by a tilting. Let*

$$[0, 1) \ni t \mapsto Z_t \in N(X)_\mathbb{C}^\vee$$

be a continuous map such that $\sigma_t = (Z_t, \mathcal{A})$ for any rational number $0 < t < 1$ and $\sigma_0 = (Z_0, \mathcal{A}')$ determine points in $\text{Stab}(X)$. Then we have $\lim_{t \rightarrow +0} \sigma_t = \sigma_0$.

4.5. Moduli spaces. Let \mathcal{M} be the algebraic space which parameterizes objects $E \in D^b \text{Coh}(X)$ satisfying

$$\text{Ext}^{<0}(E, E) = 0, \quad \text{Hom}(E, E) = \mathbb{C},$$

constructed by Inaba [14]. For $\sigma = (Z, \mathcal{A}) \in \text{Stab}(X)_\mathbb{R}$, let

$$\mathcal{M}^\sigma([\mathcal{O}_x]) \subset \mathcal{M}$$

be the subspace which parameterizes Z -stable objects $E \in \mathcal{A}$ with $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ for $x \in X$. Note that, a priori, $\mathcal{M}^\sigma([\mathcal{O}_x])$ is just

an abstract subfunctor of \mathcal{M} from the category of \mathbb{C} -schemes to the category of sets. The subspace $\mathcal{M}^\sigma([\mathcal{O}_x])$ is shown to be an algebraic subspace if the openness of σ -stable objects is proved. (See [22] for the arguments when X is a K3 surface or an abelian surface.) The following proposition completes the proof of Theorem 1.2:

Proposition 4.13. *For $\sigma \in U(Y)$, the space $\mathcal{M}^\sigma([\mathcal{O}_x])$ is an open algebraic subspace of \mathcal{M} , and isomorphic to Y .*

Proof. By deforming $\sigma \in U(Y)$, we may assume that σ is written as

$$(Z_{f^*\omega+D}, \mathcal{A}_\omega^\dagger(X/Y))$$

for some rational point $f^*\omega + D \in A^\dagger(Y)$. In order to reduce the notation, we write $Z = Z_{f^*\omega+D}$. Let us take an object $E \in \mathcal{A}_\omega^\dagger(X/Y)$, giving a \mathbb{C} -valued point of $\mathcal{M}^\sigma([\mathcal{O}_x])$. It fits into an exact sequence in $\mathcal{A}_\omega^\dagger(X/Y)$

$$0 \rightarrow F \rightarrow E \rightarrow \mathbf{L}f^*M \rightarrow 0$$

for some $F \in \mathcal{C}_{X/Y}^0$ and $M \in \mathcal{A}_\omega$. If $F \neq 0$, we have $\mathrm{Im} Z(F) > 0$, $\mathrm{Im} Z(\mathbf{L}f^*M) \geq 0$, hence $\mathrm{Im} Z(E) > 0$. This contradicts to $\mathrm{ch}(E) = \mathrm{ch}(\mathcal{O}_x)$, hence $F = 0$ and $E \cong \mathbf{L}f^*M$. Since $M \in \mathcal{A}_\omega$ satisfies $\mathrm{ch}(M) = \mathrm{ch}(\mathcal{O}_y)$, Lemma 2.7 implies that $M \cong \mathcal{O}_y$ for some $y \in Y$, i.e. $E \cong \mathbf{L}f^*\mathcal{O}_y$. Conversely, let us consider the object $\mathbf{L}f^*\mathcal{O}_y \in \mathcal{A}_\omega(X/Y)$. Since $\mathcal{O}_y \in \mathcal{A}_\omega$ is $Z_{\omega,D}$ -stable by Lemma 2.7, Lemma 4.6 implies that $\mathbf{L}f^*\mathcal{O}_y \in \mathcal{A}_\omega(X/Y)$ is also Z -stable.

The above argument shows that the morphism

$$(62) \quad Y \rightarrow \mathcal{M}$$

sending y to $\mathbf{L}f^*\mathcal{O}_y$ induces a bijection between closed points of Y and those of $\mathcal{M}^\sigma([\mathcal{O}_x])$. Also since the functor

$$\mathbf{L}f^*: D^b \mathrm{Coh}(Y) \rightarrow D^b \mathrm{Coh}(X)$$

is fully faithful, the morphism (62) is bijective on the tangent spaces. Therefore it is enough to show that $\mathcal{M}^\sigma([\mathcal{O}_x])$ is open in \mathcal{M} , which follows if we show that the objects of the form $\mathbf{L}f^*\mathcal{O}_y$ are closed under deformations.

Note that an object $E \in D^b \mathrm{Coh}(X)$ is written as $\mathbf{L}f^*M$ for some $M \in D^b \mathrm{Coh}(Y)$ if and only if

$$(63) \quad \mathrm{Hom}(\mathcal{D}_{X/Y}, E) = 0,$$

where $\mathcal{D}_{X/Y}$ is defined in Subsection 2.3. Since $\mathcal{D}_{X/Y}$ is the smallest triangulated subcategory which contains some finite number of objects in $\mathcal{D}_{X/Y}$, the condition (63) is an open condition by the upper semicontinuity. Hence if E is a small deformation of $\mathbf{L}f^*\mathcal{O}_y$, it is of the form $\mathbf{L}f^*M$ for some $M \in D^b \mathrm{Coh}(Y)$. Then M is a small deformation of \mathcal{O}_y , hence $M \cong \mathcal{O}_{y'}$ for some $y' \in Y$. The statement is now proved. \square

5. SOME TECHNICAL RESULTS

In this section, we give proofs of Proposition 2.8 and Proposition 4.8.

5.1. Proof of Proposition 2.8.

Proof. We divide the proof into four steps.

Step 1.

Let us take a rational point $\omega \in \overline{A}(X)$. By Theorem 2.3 and Proposition 2.6, there are open neighborhoods

$$(64) \quad \omega \in U_\omega \subset \text{NS}(X)_\mathbb{R}, \quad \sigma_\omega \in \mathcal{U}_\omega \subset \text{Stab}(X)_\mathbb{R}$$

such that $\Pi_\mathbb{R}$ restricts to a homeomorphism between \mathcal{U}_ω and U_ω . We claim that, after shrinking (64) if necessary, we have

$$(65) \quad \sigma_{\omega'} \in \mathcal{U}_\omega, \text{ for any rational } \omega' \in U_\omega \cap \overline{A}(X).$$

To prove this, we may assume that ω' lies in the interior $A(X) \subset \overline{A}(X)$ since $\text{Stab}(X)_\mathbb{R}$ is Hausdorff. Let us take a stability condition $\tilde{\sigma}_{\omega'} \in \mathcal{U}_\omega$ satisfying $\Pi_\mathbb{R}(\tilde{\sigma}_{\omega'}) = \omega'$. By [21, Proposition 3.14], after shrinking (64) if necessary, any object \mathcal{O}_x for $x \in X$ is $\tilde{\sigma}_{\omega'}$ -stable of phase one. Then Lemma 5.1 below shows that

$$\tilde{\sigma}_{\omega'} = \sigma_{\omega'}.$$

Therefore the condition (65) holds.

Step 2.

By the property (65) and Theorem 2.3, there is an open subset $U \subset \overline{A}(X)$, which contains all the rational points, such that the construction in Proposition 2.6 extends to a continuous map

$$\sigma_U: U \rightarrow \text{Stab}(X)_\mathbb{R}.$$

It is enough to show that σ_U extends to the whole of $\overline{A}(X)$. We first show that σ_U extends to $U \cup A(X)$. Let us take an irrational point $\omega \in A(X)$, and rational points $\omega_j \in A(X)$ for $j \geq 1$ which converge to ω . By Proposition 2.6, there is a constant $K_j > 0$ such that

$$\frac{\|E\|}{|Z_{\omega_j}(E)|} < K_j,$$

for any non-zero σ_{ω_j} -semistable object E . By the evaluation of K_j in the proof of [21, Proposition 3.13], one can easily check that the K_j is taken to be independent of j . Indeed, the evaluation of [21, Equation (33)] concerns the constant $C_{\omega_j} \geq 0$,

$$C_{\omega_j} := \sup \left\{ -\frac{D^2 \cdot \omega_j^2}{(D \cdot \omega_j)^2} : D \text{ is an effective divisor on } X \right\},$$

(cf. [4, Corollary 7.3.3],) and

$$\sup \left\{ \frac{u}{(u + \omega_j^2/2)^2} : u > 0 \right\}.$$

By the openness of $A(X)$, these values are bounded above by a constant which is independent of j . This easily implies that

$$\lim_{j \rightarrow \infty} \left\{ \frac{|Z_{\omega_j}(E) - Z_{\omega}(E)|}{|Z_{\omega_j}(E)|} : E \text{ is } \sigma_{\omega_j}\text{-semistable} \right\} = 0.$$

Therefore, by [7, Theorem 7.1], there is $\sigma_{\omega} \in \text{Stab}(X)_{\mathbb{R}}$ satisfying

$$(66) \quad \lim_{j \rightarrow \infty} \sigma_{\omega_j} = \sigma_{\omega}.$$

Step 3.

We need to show that σ_{ω} in (66) is independent of ω_j . In order to show this, we claim that \mathcal{O}_x is σ_{ω} -stable for any $x \in X$. Suppose that \mathcal{O}_x is not σ_{ω} -stable. Since ω_j is rational, $\mathcal{O}_x \in \mathcal{A}_{\omega_j}$ is σ_{ω_j} -stable by Lemma 2.7, hence \mathcal{O}_x is σ_{ω} -semistable. This implies that there is a non-trivial σ_{ω} -stable factor A of \mathcal{O}_x , and ω is a solution of $\omega \cdot c_1(A) = 0$.

On the other hand, let us take a sufficiently small open neighborhood $\sigma_{\omega} \in \mathcal{U}_{\omega}$. Since σ_{ω} satisfies the support property, there is a wall and chamber structure on \mathcal{U}_{ω} with finite number of codimension one walls such that the set of semistable objects E with $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ is constant at a chamber but jumps at a wall. By the argument as above, σ_{ω} lies at the wall of the form $\Pi_{\mathbb{R}}(*) \cdot c_1(A) = 0$. Since the image of this wall under $\Pi_{\mathbb{R}}$ contains dense rational points, we can deform σ_{ω} to $\tilde{\sigma}_{\omega''}$ on the wall such that its image under $\Pi_{\mathbb{R}}$ is a rational point $\omega'' \in A(X)$. Since $\tilde{\sigma}_{\omega''}$ lies at the wall, it is a limit of stability conditions of the form $\sigma_{\omega_j''}$ for $j \geq 1$ with ω_j'' rational and $\omega_j'' \rightarrow \omega''$. However, by the property (65), the stability condition $\sigma_{\omega''}$ is also the limit of $\sigma_{\omega_j''}$. Therefore $\tilde{\sigma}_{\omega''} = \sigma_{\omega''}$, which is a contradiction since \mathcal{O}_x is not $\tilde{\sigma}_{\omega''}$ -stable but $\sigma_{\omega''}$ -stable. Therefore \mathcal{O}_x is σ_{ω} -stable,

Since \mathcal{O}_x is σ_{ω} -stable, if we take open subsets as in (64) for an irrational ω , then the same argument as in Step 1 shows that they satisfy the condition (65). This immediately implies that σ_{ω} is independent of the choice of ω_j . Hence σ_U extends to the continuous map from $U \cup A(X)$, by sending ω to σ_{ω} .

Step 4.

The final step is to extend the map from $U \cup A(X)$ to the map from $\overline{A}(X)$. Let us take an irrational point $\omega \in \overline{A}(X) \setminus A(X)$, and rational points $\omega_j \in \overline{A}(X) \setminus A(X)$ for $j \geq 1$ which converge to ω . By the same reason as in Step 2, the limit σ_{ω} of σ_{ω_j} exists. Note that $A(X)$ is continuously embedded into $\text{Stab}(X)_{\mathbb{R}}$ by the previous step, which gives a section of $\Pi_{\mathbb{R}}$ over $A(X)$. Since $\sigma_{\omega_j}, \sigma_{\omega}$ lie its boundary, σ_{ω} is

uniquely determined by ω if it exists, and independent of the choice of ω_j . Now the assignment $\omega \mapsto \sigma_\omega$ gives the desired continuous map (13). \square

We have used the following lemma, which is essentially proved in [8].

Lemma 5.1. *Let $\mathcal{A} \subset D^b \text{Coh}(X)$ be the heart of a bounded t -structure and $\omega \in \text{NS}(X)_\mathbb{R}$ is ample. Suppose that the following conditions hold:*

- *The pair (Z_ω, \mathcal{A}) is a stability condition on $D^b \text{Coh}(X)$.*
- *For any $x \in X$, we have $\mathcal{O}_x \in \mathcal{A}$, and it is Z_ω -stable.*

Then we have $\mathcal{A} = \mathcal{A}_\omega$.

Proof. The result is essentially proved in [8, Proposition 10.3, Step 2], using [8, Lemma 10.1]. Although these results in [8] are stated for K3 surfaces or abelian surfaces, one can see that the arguments work for arbitrary projective surfaces. \square

5.2. Proof of Proposition 4.8.

Proof. The proof is similar to that of Proposition 2.8, but we need to take more care because we are no longer able to use Lemma 5.1.

Step 1.

Let us take a rational point $f^*\omega + D \in \overline{A}^\dagger(Y)$. By Theorem 2.3 and Proposition 2.6, there are open neighborhoods

$$(67) \quad f^*\omega + D \in U_{\omega,D} \subset \text{NS}(X)_\mathbb{R},$$

$$(68) \quad \sigma_{f^*\omega+D} \in \mathcal{U}_{\omega,D} \subset \text{Stab}(X)_\mathbb{R}$$

such that $\Pi_\mathbb{R}$ restricts to a homeomorphism between $\mathcal{U}_{\omega,D}$ and $U_{\omega,D}$. We claim that, after shrinking (67), (68) if necessary, we have

$$(69) \quad \sigma_{f^*\omega'+D'} \in \mathcal{U}_{\omega,D}, \text{ for any rational } f^*\omega' + D' \in U_{\omega,D} \cap \overline{A}^\dagger(Y).$$

Let us take $\tilde{\sigma}_{f^*\omega'+D'} \in \mathcal{U}_{\omega,D}$ whose image under $\Pi_\mathbb{R}$ is $f^*\omega' + D'$. It is enough to show

$$(70) \quad \tilde{\sigma}_{f^*\omega'+D'} = \sigma_{f^*\omega'+D'}.$$

Step 2.

Below, we assume that the reader is familiar with the notion of slicings in the original paper [7], and the related notations. The stability conditions $\sigma_{f^*\omega+D}$, $\tilde{\sigma}_{f^*\omega'+D'}$ are written as pairs

$$\sigma_{f^*\omega+D} = (Z_{f^*\omega+D}, \mathcal{P}),$$

$$\tilde{\sigma}_{f^*\omega'+D'} = (Z_{f^*\omega'+D'}, \mathcal{P}'),$$

for slicings $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$, $\mathcal{P}' = \{\mathcal{P}'(\phi)\}_{\phi \in \mathbb{R}}$. We also consider stability conditions on $D^b \text{Coh}(Y)$

$$\sigma_{\omega,D} = (Z_{\omega,D}, \mathcal{A}_\omega),$$

$$\sigma_{\omega',D'} = (Z_{\omega',D'}, \mathcal{A}_{\omega'})$$

considered in Lemma 4.4. We denote by

$$\{\mathcal{Q}(\phi)\}_{\phi \in \mathbb{R}}, \{\mathcal{Q}'(\phi)\}_{\phi \in \mathbb{R}}$$

the slicings corresponding to $\sigma_{\omega, D}$, $\sigma_{\omega', D'}$ respectively.

By shrinking (67), (68) if necessary, there is $0 < \epsilon < 1/8$ so that

$$d(\mathcal{P}, \mathcal{P}') < \epsilon,$$

where $d(*, *)$ is given in [7, Section 6]. Then the category $\mathcal{P}'(\phi)$ is the category of $Z_{f^*\omega' + D'}$ -semistable object in the quasi-abelian category (cf. the proof of [7, Theorem 7.1])

$$(71) \quad \mathcal{P}((\phi - \epsilon, \phi + \epsilon)).$$

Also, similarly to the proof of Proposition 5.1, Step 1, we see that $\sigma_{\omega', D'}$ is contained in an open neighborhood of $\sigma_{\omega, D}$. Consequently, we may assume that $d(\mathcal{Q}, \mathcal{Q}') < \epsilon$, and $\mathcal{Q}'(\phi)$ is the category of $Z_{\omega', D'}$ -semistable objects in the quasi-abelian category

$$\mathcal{Q}((\phi - \epsilon, \phi + \epsilon)).$$

Step 3.

The relation (70) follows if we show

$$(72) \quad \mathcal{A}_{f^*\omega'}^\dagger \subset \mathcal{P}'((0, 1])$$

since both sides are hearts of bounded t-structures on $D^b \text{Coh}(X)$. The inclusion (72) is equivalent to

$$(73) \quad \mathcal{C}_{X/Y}^0 \subset \mathcal{P}'((0, 1]),$$

$$(74) \quad \mathbf{L}f^* \mathcal{A}_{\omega'} \subset \mathcal{P}'((0, 1]).$$

We first prove (73). Since $\mathcal{C}_{X/Y}^0$ is the extension closure of a finite number of objects, and $\text{Im } Z_D(\mathcal{C}_{X/Y}^0 \setminus \{0\}) > 0$, we have

$$(75) \quad \mathcal{C}_{X/Y}^0 \subset \mathcal{P}([\theta, \theta'])$$

for some $\theta, \theta' \in (0, \pi)$. By shrinking (67), (68) if necessary, we may assume that

$$(76) \quad [\theta - \epsilon, \theta' + \epsilon] \subset (0, \pi).$$

Then the condition (73) follows since $d(\mathcal{P}, \mathcal{P}') < \epsilon$.

The inclusion (74) follows if we show $\mathbf{L}f^* \mathcal{Q}'(\phi) \subset \mathcal{P}'(\phi)$ for any $0 < \phi \leq 1$. By the argument in Step 2, it is enough to show that

$$(77) \quad \mathbf{L}f^* \mathcal{Q}((\phi - \epsilon, \phi + \epsilon)) \subset \mathcal{P}((\phi - \epsilon, \phi + \epsilon))$$

and for any $M \in \mathcal{Q}((\phi - \epsilon, \phi + \epsilon))$ and an exact sequence in $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$

$$(78) \quad 0 \rightarrow F_1 \rightarrow \mathbf{L}f^* M \rightarrow F_2 \rightarrow 0,$$

we have

$$(79) \quad F_i \in \mathbf{L}f^* \mathcal{Q}((\phi - \epsilon, \phi + \epsilon)).$$

Step 4.

By Lemma 4.6, we have

$$(80) \quad \mathbf{L}f^*\mathcal{Q}(\psi) \subset \mathcal{P}(\psi)$$

for any $\psi \in \mathbb{R}$. Then the inclusion (77) is obvious from (80). Suppose that there is an exact sequence (78) in $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$. If $\phi \in (\epsilon, 1 - \epsilon)$, we have $\mathcal{P}((\phi - \epsilon, \phi + \epsilon)) \subset \mathcal{A}_\omega(X/Y)$. By Lemma 4.2, it follows that $F_i \in \mathbf{L}f^*\mathcal{A}_\omega$ for $i = 1, 2$. Together with $F_i \in \mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ and the condition (80), we conclude that (79) holds.

If $\phi \notin (\epsilon, 1 - \epsilon)$, we have either $\phi \in (0, \epsilon)$ or $\phi \in (1 - \epsilon, 1]$. These cases are treated similarly, so we assume $\phi \in (1 - \epsilon, 1]$ for simplicity. By setting $\mathcal{A} = \mathcal{A}_\omega(X/Y)$, we have the exact sequence in \mathcal{A}

$$(81) \quad \begin{aligned} 0 &\rightarrow \mathcal{H}_{\mathcal{A}}^{-1}(F_1) \rightarrow \mathcal{H}_{\mathcal{A}}^{-1}(\mathbf{L}f^*M) \rightarrow \mathcal{H}_{\mathcal{A}}^{-1}(F_2) \\ &\rightarrow \mathcal{H}_{\mathcal{A}}^0(F_1) \rightarrow \mathcal{H}_{\mathcal{A}}^0(\mathbf{L}f^*M) \rightarrow \mathcal{H}_{\mathcal{A}}^0(F_2) \rightarrow 0. \end{aligned}$$

Since $\mathbf{L}f^*\mathcal{A}_\omega \subset \mathcal{A}_\omega(X/Y)$, we have

$$\mathcal{H}_{\mathcal{A}}^i(\mathbf{L}f^*M) \cong \mathbf{L}f^*\mathcal{H}_{\mathcal{A}_\omega}^i(M),$$

for all i . By Lemma 4.2 and the exact sequence (81), we have

$$\mathcal{H}_{\mathcal{A}}^{-1}(F_1), \mathcal{H}_{\mathcal{A}}^0(F_2) \in \mathbf{L}f^*\mathcal{A}_\omega.$$

By the condition (76), we have

$$\mathrm{Hom}(\mathcal{C}_{X/Y}^0, \mathcal{H}_{\mathcal{A}}^{-1}(F_2)) = 0$$

since $\mathcal{H}_{\mathcal{A}}^{-1}(F_2) \in \mathcal{P}((0, \epsilon))$. This implies that $\mathcal{H}_{\mathcal{A}}^{-1}(F_2) \in \mathbf{L}f^*\mathcal{A}_\omega$, hence $\mathcal{H}_{\mathcal{A}}^0(F_1) \in \mathbf{L}f^*\mathcal{A}_\omega$ also holds by Lemma 4.2 and the exact sequence (81).

We have shown that $\mathcal{H}_{\mathcal{A}}^j(F_i)$ is an object in $\mathbf{L}f^*\mathcal{A}_\omega$ for all i and j . Since the functor

$$\mathbf{L}f^*: D^b \mathrm{Coh}(Y) \rightarrow D^b \mathrm{Coh}(X)$$

is fully faithful, it follows that $F_i \in \mathbf{L}f^*D^b \mathrm{Coh}(Y)$ for $i = 1, 2$. Combined with (80), we conclude that (79) holds.

Step 5.

Now we have proved (69). By the property (69), there is an open subset $U_Y \subset \overline{A}^\dagger(Y)$, which contains all the rational points, such that the construction in Proposition 4.7 extends to a continuous map

$$\sigma_{U,Y}: U_Y \rightarrow \mathrm{Stab}(X)_{\mathbb{R}}.$$

We next show that $\sigma_{U,Y}$ extends to the $U_Y \cup A^\dagger(Y)$. For an irrational point $f^*\omega + D \in A^\dagger(Y)$, let us take rational points $f^*\omega_j + D_j \in A^\dagger(Y)$

for $j \geq 1$ which converge to $f^*\omega + D$. By Lemma 4.5, there is a constant $K_j > 0$ so that

$$\frac{\|E\|}{|Z_{f^*\omega_j + D_j}(E)|} < K_j$$

for any non-zero $\sigma_{f^*\omega_j + D_j}$ -semistable object E . By the evaluation of K_j in the proof of Proposition 4.7, and the argument in the proof of Proposition 2.8, Step 2, it is easy to see that the constant K_j is taken to be independent of j . Therefore, as in the proof of Proposition 2.8, Step 2, the limit exists

$$(82) \quad \sigma_{f^*\omega + D} := \lim_{j \rightarrow \infty} \sigma_{f^*\omega_j + D_j}.$$

Step 6.

We claim that the limit (82) does not depend on a choice of $f^*\omega_j + D_j$. Indeed if we take another rational points $f^*\omega'_j + D'_j \in A^\dagger(Y)$ which converge to $f^*\omega + D$, then Theorem 2.3 implies the existence of

$$\tilde{\sigma}_{f^*\omega'_j + D'_j} \in \text{Stab}(X)_{\mathbb{R}}$$

for $j \gg 0$ whose image under $\Pi_{\mathbb{R}}$ is $f^*\omega'_j + D'_j$ and converge to $\sigma_{f^*\omega + D}$. Let us write

$$\begin{aligned} \sigma_{f^*\omega_j + D_j} &= (Z_{f^*\omega_j + D_j}, \mathcal{P}_j), \\ \tilde{\sigma}_{f^*\omega'_j + D'_j} &= (Z_{f^*\omega'_j + D'_j}, \mathcal{P}'_j), \end{aligned}$$

for slicings $\mathcal{P}_j = \{\mathcal{P}_j(\phi)\}_{\phi \in \mathbb{R}}$ and $\mathcal{P}'_j = \{\mathcal{P}'_j(\phi)\}_{\phi \in \mathbb{R}}$. Then $d(\mathcal{P}_j, \mathcal{P}'_j)$ goes to zero for $j \rightarrow \infty$. Also we can take $\theta, \theta' \in (0, \pi)$, which does not depend on j , so that

$$\mathcal{C}_{X/Y}^0 \subset \mathcal{P}_j([\theta, \theta'])$$

for all $j \gg 0$. If we take $0 < \epsilon < 1/8$ satisfying (76), we have

$$(83) \quad \mathcal{C}_{X/Y}^0 \subset \mathcal{P}'_j((0, 1]),$$

for all $j \gg 0$ satisfying $d(\mathcal{P}_j, \mathcal{P}'_j) < \epsilon$. Also, by the same argument of Proposition 2.8, Step 3, one sees that the stability conditions σ_{ω_j, D_j} and $\sigma_{\omega'_j, D'_j}$ converge to the same point in $\text{Stab}(Y)_{\mathbb{R}}$. Using this fact instead of the two sentences after (71), the same argument proving (74) shows the inclusion,

$$(84) \quad \mathbf{L}f^*\mathcal{A}_{\omega'_j} \subset \mathcal{P}'_j((0, 1]).$$

The inclusions (83), (84) imply $\mathcal{P}'_j((0, 1]) = \mathcal{A}_{f^*\omega'_j}^\dagger$ for $j \gg 0$, which implies

$$\tilde{\sigma}_{f^*\omega'_j + D'_j} = \sigma_{f^*\omega'_j + D'_j}, \quad j \gg 0.$$

Hence $\sigma_{f^*\omega + D}$ is independent of $f^*\omega'_j + D'_j$, and the assignment $f^*\omega + D \mapsto \sigma_{f^*\omega + D}$ gives a continuous map from $U_Y \cup A^\dagger(Y)$.

Step 7.

We finally extend the map from $U_Y \cup A^\dagger(Y)$ to the map from $\overline{A}^\dagger(Y)$. Let us take an irrational point $f^*\omega + D \in \overline{A}^\dagger(Y) \setminus A^\dagger(Y)$, and rational points $f^*\omega_j + D_j \in \overline{A}^\dagger(Y) \setminus A^\dagger(Y)$ for $j \geq 1$ which converge to $f^*\omega + D$. Similarly to the argument of Step 5, the limit $\sigma_{f^*\omega+D}$ of $\sigma_{f^*\omega_j+D_j}$ exists. Since we have shown that $A^\dagger(Y)$ is continuously embedded into $\text{Stab}(X)_\mathbb{R}$, the same argument in the proof of Proposition 5.1, Step 4 shows that $\sigma_{f^*\omega+D}$ is independent of $f^*\omega_j + D_j$. Now the assignment $f^*\omega + D \mapsto \sigma_{f^*\omega+D}$ gives the desired continuous map (53). \square

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Todai Institute for Advanced Studies (TODIAS),
Kavli Institute for the Physics and Mathematics of the Universe,
University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.
E-mail address: yukinobu.toda@ipmu.jp